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CLOSURE THEOREMS FOR SPACES OF ENTIRE FUNCTIONS

BY LOREN D. PITT[1]

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We announce a number of single variable approximation theorems. Our approach is to extend de Branges' basic theory of Hilbert spaces of entire functions [2] to a Banach space setting. The resulting structure is sufficiently rich to provide both new approximation results and a unifying structure for many earlier results on approximation by entire functions which are related to the Bernstein approximation problem, for example, Akutowitz [1], Koosis [3], Levinson and McKean [5], Mergelyan [6], Pitt [7] and Pollard [8].

Let C_c be the space of continuous complex functions $m(\lambda)$ on R^1 with compact support and the supremum norm $|m|$. B denotes a fixed Banach function space on R^1 with (semi) norm $\|f\|$. We assume that

(1) $C_c \cap B$ is dense in B, and

(2) The multiplication operator $(m, f) \longrightarrow m(\lambda)f(\lambda)$ is jointly continuous from $C_c \times B$ into B.

Examples of spaces satisfying (1) and (2) are L^p spaces, Orlicz spaces, Lorentz spaces $L_{(p,q)}$ and spaces of continuous functions with weighted supremum norms. Because of condition (2) it follows that for $f \in B$ and $e \in B^*$, the linear functional on C_c given by $m \longrightarrow \langle mf, e\rangle$ is expressible in the form $\langle mf, e\rangle = \int m(\lambda)d\mu_{f,e}$ where $\mu_{f,e}$ is a unique finite Radon measure. The discrete spectrum $\sigma_d(B)$ of B is the set $\{\lambda : |\mu_{f,e}\{\lambda\}| > 0 \text{ for some } f \in B \text{ and } e \in B^*\}$.

Contained in B we fix a linear space H of entire functions with closure $\overline{H}$. We assume for $\operatorname{Im} z \neq 0$ and for f and g in H that the function

$$(3) \qquad F(\lambda) \equiv (z - \lambda)^{-1}\{f(z)g(\lambda) - g(z)f(\lambda)\} \in H.$$

If H is closed under the conjugation $h \longrightarrow \overline{h}(\overline{z})$ we call H symmetric. Two basic examples of symmetric H are the space $\mathcal{P}$ of all polynomials and the space $\mathcal{F}(T)$

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of all Fourier transforms $f(z) = \int \exp\{itz\}g(t)\,dt$ where g is infinitely differentiable and supported on $[-T, T]$. To avoid trivialities we assume

(4) for each $\lambda \in \sigma_d(B)$ there exists an $h \in H$ with $h(\lambda) \neq 0$.

The statement of our results require the auxiliary norm $\|f\|_+ = \|(\lambda - i)^{-1} \cdot f(\lambda)\|$ and the evaluation functionals $\{e_z; z \in \mathbf{C}'\}$ on H where $e_z(f) = f(z)$ together with the norms $L(z) = \|e_z\|$ and $L^+(z) = \|e_z\|_+$.

THEOREM 1. *If $L^+(\beta) = +\infty$ for some $\beta \in R^{2+} = \{z\colon \operatorname{Im} z > 0\}$ then*

$$(z - \lambda)^{-1}\bar{H} \subseteq \bar{H} \quad \text{for each } z \in R^{2+}.$$

THEOREM 2. *Let $H_\beta = \{h \in H\colon h(\beta) = 0\}$. If H is symmetric and if $(\beta - \lambda)^{-1}H_\beta$ is dense in H for some β with $\operatorname{Im}(\beta) \neq 0$ then $\bar{H} = B$ iff $L^+(\beta) = +\infty$.*

THEOREM 3. *If $\beta \in R^{2+}$ and $0 < L(\beta) < \infty$ then $L(z)$ is continuous and subharmonic on R^{2+}. If in addition $0 < L(\gamma) < \infty$ for some $\gamma \in R^{2-}$ then $L(z)$ is continuous and subharmonic on $\mathbf{C}^1$.*

Under the conditions of Theorem 3, $\bar{H}$ is a closed subspace of entire functions $f(z)$ satisfying $|f(z)| \leqslant L(z)\|f\|$ and (3).

THEOREM 4. *Assume H is closed and that $L(z)$ is finite. Let $K = B \cap \{f\colon f(z)$ is entire and $f(z)L^{-1}(z)$ is bounded on $\mathbf{C}^1\}$. Then $H \subseteq K$ and the codimension of H in K satisfies $\dim(K|H) \leqslant 1$.*

The case $K = H$ is generic but $\dim(K|H) = 1$ can occur.

THEOREM 5. *Under the conditions of Theorem 4 there exist functions h_+ and h_- in H for which K consists of all entire functions $f \in B$ satisfying*

(i) *$f(z)h_+^{-1}(z)$ (resp. $f(z)h_-^{-1}(z)$) is analytic and of bounded type on R^{2+} (resp. R^{2-}).*

(ii) $\sup f(iy)L^{-1}(iy) < \infty$, $y \in R^1$.

These theorems can be refined when $H = P$ or $H = F(T)$. The solutions of the Bernstein problem given in [1], [6], [8] are generalized to the present setting by

THEOREM 6. *If $H = P$ or $H = F(T)$ then $\bar{H} = B$ iff either of the equivalent conditions*

(i) $L^+(i) = +\infty$,

(ii) $\int \log L^+(\lambda)(1 + \lambda^2)^{-1}\,d\lambda = +\infty$,

is satisfied.

The generalizations of the Paley-Wiener theorem given in [1], [4], [5] also hold in the present case. We set $\bar{F}(T+) = \bigcap\{\bar{F}(S)\colon S > T\}$.

THEOREM 7. *Either $\bar{F}(T+) = B$ or $\bar{F}(T+) = B \cap E(T)$, where $E(T)$ is* *the space of entire functions of exponential type not greater than T.*

When B is a classical sequence space it may happen that both $L(z) < \infty$ and $H = B$. Series expansions for $L(z)$ are possible in this case and results related to classical interpolatory function theory may be obtained (see [7, p. 115]).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903