are in order. The book is not primarily intended to treat the applications of differentiation theory to real and complex analysis. Rather, for the most part, the author tries to achieve great depth in treating the "pure" differentiation theory itself. He therefore provides a background for the material discussed here rather than this material proper. De Guzmán's book contains the basic Lebesgue theorem on differentiation of the integral and the Hardy-Littlewood maximal theorem along with a great many variants of these theorems, proven by the use of covering lemmas. The variants of the Vitali lemma which the author treats are also quite numerous. It is extremely commendable that the Calderón-Zygmund decomposition is proven, and the disk multiplier problem is mentioned, with a few words about C. Fefferman's solution to the problem. In addition, De Guzmán includes a careful treatment of differentiation theory with respect to two other extremely crucial differentiation bases besides the class of balls (or what is essentially the same thing, cubes). These are the bases of all rectangles in R^n with sides parallel to the coordinate axes, and the larger class of all rectangles in R^n with arbitrary orientation. The relationship between covering lemmas, maximal theorems, and differentiation theorems is also discussed. These important topics, as well as many others make the book's content worthwhile for the experts of this subject or for students who would like to learn these areas, and then branch out by studying the important applications.

De Guzmán's book is carefully written, and the style makes for easy and enjoyable reading. His work is a significant contribution to the field which should be welcomed by all concerned with this beautiful area of mathematics.

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Invariants for real-generated uniform topological and algebraic categories, by Kevin A. Broughan, Lecture Notes in Mathematics, No. 491, Springer-Verlag, Berlin, Heidelberg, New York, 1975, x + 197 pp., \$8.20.

The literature on relations between the dimension of a metrizable space X and the existence of metrics on X having convenient special properties is rather extensive (see Nagata's book [5] for the only good exposition) and contains two really successful theorems. First, Hausdorff formalized the idea of estimating the measure of a set A in t-dimensional space by covering it with finitely many ε_i-spheres and taking their measures to be ε_i^t. It turns out (L. Pontrjagin and L. Schnirelmann, 1932; E. Szpilrajn, 1937; book [5, pp. 112–116]) that the dimension of separable metrizable A is the infimum of the real numbers t such that for some metric on A, the t-dimensional measure is 0. The second theorem is P. Ostrand's [6] (improving results of J. de Groot, 1957, and J. I. Nagata, 1958; book [5, pp. 137–154]): metrizable X has covering dimension $\leqslant n$ if and only if it has a metric in which, for all ε, given

$n + 2$ points y_i in the ε-neighborhood of a point x, some two y's are within ε of each other.

In Ostrand's theorem one must specify covering dimension, dim X, because it is in general greater than small inductive dimension ind X. In separable spaces they coincide, and the t-dimensional measure idea depends on (at most) countable sums and cannot be used for nonseparable spaces. By definition dim $X \leqslant n$ when every finite open cover has a refinement no $n + 2$ of whose members have a common point; ind $X \leqslant n + 1$ when each point has a basis of neighborhoods with at most n-dimensional boundaries (respectively, ind $X = 0 \cdots$, with empty boundaries). There are natural modifications of both dim and ind: large covering dimension, given by arbitrary open coverings, and large inductive dimension, given by neighborhoods of closed sets instead of points. But in this setting, metrizable topological spaces, both these notions coincide with dim.

Among several similar special results on zero-dimensional metric spaces, the most interesting is Broughan's [**1**]: metrizable X admits a metric d whose set of values is contained in the union of $\{0\}$ and the set of unit fractions $1/n$, if and only if dim $X = 0$. (It should be noted that the proof is hard, but the hard part is earlier work of K. Morita [**4**].) More generally, if X has a metric d whose set of values $d(X^2) \subset [0, \infty)$ is not a neighborhood of 0, plainly

$$\operatorname{ind} X = 0.$$

So in the separable case the result is that X has a metric with $d(X^2)$ zero-dimensional if and only if X is zero-dimensional.

The first problem is whether every metric space d for which $d(X^2)$ is not a neighborhood of zero satisfies dim $X = 0$. In other words, classifying subsets S of $[0, \infty)$ by equality of the classes $M(S)$ of all metrizable topological spaces admitting an S-valued metric, is there only one nontrivial class? Broughan shows that all S in which 0 is a nonisolated point, but whose closure is not a neighborhood of 0, are equivalent. The next question would seem to be, if X has (say) a rational-valued metric, is every closed set the intersection of a simple sequence of clopen sets?

The second problem concerns the classification of these sets S by the classes $U(S)$ of S-metrizable uniform spaces. Here there is certainly more than one nontrivial class, and the problem is, are there more than two? One class is given by precisely those S just mentioned, containing 0 as a nonisolated point but not dense in any neighborhood of 0; and it is the class of zero-dimensional metrizable uniform spaces. (While uniform dimension theory seems generally more confused than topological, it is the opposite for zero-dimensional metric spaces: all the usual uniform dimension functions coincide [**3**]. This greater simplicity may prevail more generally; the uniform theory is short of both theorems and counterexamples.) If S is dense in a neighborhood of zero, the space Q of rationals is (uniformly) S-metrizable, though its uniform dimension is 1. However, one can see by examining the completion that the uniform space of irrationals has no countable-valued metric. Characterizing the class of Q^+-metrizable uniform spaces is a tricky problem: an aspect, certainly, of the main problem.

Broughan's book includes a formal characterization of the Q^+-metrizable uniform spaces, but it is just the existence of an indexed family of entourages of the same order type as Q and so structured as to give the metric at once.

The third possibly very interesting problem concerns metrics on general spaces which are rational-valued on a dense subspace. Broughan conjectures that every metrizable topological space has such a metric. Of course this is true for separable spaces, and this might be followed up to nice-valued metrics on n-dimensional separable spaces. Broughan proves that every metric space has a dense subspace D with $\dim D = 0$. This is quite easy; the author uses a sophisticated 1967 theorem of N. Kimura needlessly, for the original theorem of Čech [**2**] applies, and in a special case much easier to prove than the general case. Consequently it is not beyond human capacity to study the construction further and perhaps get a rational-valued metric on D which would extend over the whole space.

In the weird variety of topics taken up in this book (under an awkward notation and terminology, systematically invoking categories which really serve only as collections of objects–category of metrizable spaces, category of semimetric spaces, etc.), two other passages are of interest. Paralleling the notions of strong paracompactness, pointwise paracompactness, and others, the author defines clopen-paracompactness: every clopen cover has a clopen locally finite refinement. (He says "star-finite", but it is obviously equivalent.) There are several quite easy results, one pretty one: X is clopen-paracompact if and only if to every clopen cover of X there is subordinated a harmonic partition of unity (i.e. a partition into continuous functions whose values are 0 and unit fractions). And one result motivates the concept: a paracompact space X, has large inductive dimension (= dim, in the metrizable case) zero if and only if it is clopen-paracompact and $\operatorname{ind} X = 0$.

More useful, probably, is Broughan's study of six or eight types of special metric or norm on topological groups, fields, commutative rings, and vector spaces over classical disconnected fields. The usual conclusion is that among the S-metrizability properties, Q^+-metrizability is next strongest after harmonic metrizability.

At least two unsolved problems posed in this book can be solved at once by anyone conversant with (respectively) Kelley's or Sierpiński's introductory topology text. Conjecture 5.25 is that every metrizable space admits a compatible complete uniformity, which is true. Conjecture 5.23 is refuted by the observation that every complete dense-in-itself separable metric space X has a dense subset homeomorphic with the space P of irrationals. To prove it, take a homeomorphism between countable dense subsets of X and of P (this book shows how) and apply Lavrentiev's theorem: a partial homeomorphism between complete metric spaces extends to a homeomorphism of two G_δ subsets.

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Dimension theory of general spaces, by A. R. Pears, Cambridge University Press, London, New York, Melbourne, Cambridge, 1975, xii + 428 pp., \$47.50.

Interest in making the concept of dimension mathematically rigorous probably began in 1890 with the appearance of an example due to Peano of a continuous map of the unit interval onto a triangle and its interior. This created the uneasy possibility that perhaps two Euclidean spaces of different dimensions might be homeomorphic. It is hard to imagine what mathematics might have been if this had turned out to be the case. It was a close call! Fortunately, L. E. J. Brouwer gave a proof in 1911 that if R^n and R^m are homeomorphic, then $n = m$. However, it was not until the 1920's that a topological theory of dimension began to be developed. The work of K. Menger and P. Urysohn as well as others brought into existence an elegant theory of dimension applicable to all separable metric spaces. It was only incidental to this theory that Euclidean n-space was n-dimensional. In true mathematical tradition, if the unthinkable had happened, dimension theory would have continued with the same fervor. The force of mathematical inquiry would have developed a mathematical structure similar to what we have today, except for the unfortunate footnote that Euclidean n-space is not n-dimensional! Mathematics would have suffered, but not dimension theory.

In 1928 the first text in dimension theory appeared, *Dimensionstheorie* by K. Menger. This book has historical value. It reveals at one and the same time the naïveté of the early investigators by modern standards and yet their remarkable perception of what the important results were and the future direction of the theory. Copies are difficult to obtain now, but it is worth the effort.