points. Those which are not singular, are *regular*. The regular moving points are the union of disjoint annuli; the singular moving points are what is left of the moving points. The description of the flow then is given in terms of the regular moving points, where it can be completely described in terms of simple "patches"–certain standard annular flows; the fixed points–where it is, of course, fixed; and the singular moving points–where life gets complicated.

To develop the theory for singular moving points, the author first considers a very special case: the case of flows in a multiply-connected region of the plane where every orbit is aperiodic and has all its endpoints in the boundary. These he calls *Kaplan-Markus* flows, after the two who first began development of the theory of such flows. Complete success in describing such flows has eluded the author and his coworkers except, to some extent, where the set of singular fixed points has only finitely many components. The later is, however, basic, and so for many cases, another patch in the quilt yields to description. After this, the author explores various ways to combine such flows with regular flows and develop further theory.

Flows without stagnation points can be described rather completely. For flows with a finite number of stagnation points, or whose set of stagnation points has countable closure, a considerable amount is known, though the information is not as complete as for the no stagnation point case. In all cases where a description is possible, it is the set of regular moving points that supplies the main body of information. However, the author and his coworkers have developed a great deal of information about the singular moving points, forming therewith *organs* of the flow, which in turn are decomposed into *tissues* and *gametes*, and these in their turn are decomposed into *cells*. These "cells," "gametes," and "tissues" are pretty well characterized, and even the "organs" are subject to a good deal of description.

Some additional concepts have been studied, such as the algebra of flows introduced by J. and M. Lewin. One could say that, as of 1975, it is the *complete book about flows in the plane*. It is accessible to anyone with a minimal background in analysis and point set topology–provided one sticks to it sufficiently to keep track of all the terminology and notation peculiar to the book. It is light reading (except for that)–yet builds a substantial theory. It is well written and enjoyable.

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*Gaussian measures in Banach spaces*, by Hui-Hsiung Kuo, Lecture Notes in Math., no. 463, Springer-Verlag, Berlin, Heidelberg, New York, 1975, vi + 224 pp., \$9.90.

There are difficulties in constructing measures on infinite dimensional spaces. Even in a separable infinite dimensional Hilbert space the unit ball is not compact. Therefore a countably additive measure on such a space cannot be rotationally invariant . Already, in this simple space, the most natural and desirable property of Lebesgue measure has to be abandoned.

A Gaussian measure on $R^n$ can be defined by requiring that the linear functionals on $R^n$ are normally distributed. If $f$, a linear functional on $R^n$, is normally distributed with mean zero and variance $|f|^2$, where $|\ |$ is the ordinary Euclidean norm on $R^{n^*} = R^n$, the measure induced on $R^n$ is the standard, or canonical, rotationally invariant normal distribution on $R^n$. This approach cannot extend directly to Banach spaces; nevertheless it plays an important role in constructing Gaussian measures on Banach spaces.

Let $L$ be a locally convex linear topological space and $L^*$ its topological dual. A weak distribution on $L$ is an equivalence class of linear maps $F$ from $L^*$ to the linear space of random variables on some probability space. Two maps $F_1$ and $F_2$ are equivalent if for any finite set $y_1, \ldots, y_n \in L^*$ the joint distribution of $F_j(y_1), \ldots, F_j(y_n)$ is the same for $j = 1, 2$. If $L = H$ is a separable Hilbert space, a weak distribution $F$ is called a canonical normal distribution on $H$ if to each $h \in H^*$ the real valued random variable $F(h)$ is normally distributed with mean 0 and variance $\|h\|^2$. ($\|h\|$ denotes the $H^*$-norm.)

We will give an example which shows the relationship between a Gaussian measure on $H$ and a canonical normal distribution on a certain subset of $H$. We begin with the construction of a specific, natural, Gaussian measure on $H$. Let $x = \Sigma\langle x, e_n\rangle e_n$ where $\{e_n\}$ is a complete orthonormal set in the separable Hilbert space $\{H, \langle\ ,\ \rangle\}$. We define a measure $\mu$ on the cylinder sets of $H$ in terms of the joint distributions of the elements of $H^*$. Let $\Sigma\alpha_n^2 = 1$ and

$$B_{n_1,\ldots,n_k} = \left\{x \in H\colon \left(\langle x, e_{n_1}\rangle, \ldots, \langle x, e_{n_k}\rangle\right) \in B_{n_1} \times \cdots \times B_{n_k}\right\}$$

for $B_{n_1}, \ldots, B_{n_k}$ Borel sets in $R$; then

$$\mu\left(B_{n_1,\ldots,n_k}\right) = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}\,\alpha_{n_i}} \int_{B_{n_i}} \exp\left(-u^2/2\alpha_{n_i}^2\right) du. \tag{1}$$

The measure $\mu$ can be uniquely extended to a countably additive Gaussian probability measure on $H$. Clearly $\mu$ is not rotationally invariant on $H$. Let

$$H_0 = \left\{x\colon \sum \frac{\langle x, e_n\rangle^2}{\alpha_n^2} < \infty\right\}.$$

$H_0$ is a Hilbert space under the inner product

$$\langle x, y\rangle_0 = \sum \frac{\langle x, e_n\rangle\langle y, e_n\rangle}{\alpha_n^2}.$$

Let $U$ be a bounded linear operator on $H$ and $U_0 = U|_{H_0}$. Suppose that $U_0$ is a unitary operator on $\{H_0, \langle\ ,\ \rangle_0\}$. Then one can show that $\mu U^{-1} = \mu$, that is, $\mu$ is rotationally invariant with respect to rotations of $H_0$. In fact a canonical normal distribution defined on $H_0$ can be used to construct $\mu$ on $H$.

Let $F$ be a canonical normal distribution on $\{H_0, \langle\ ,\ \rangle_0\}$. Let $(\xi_1, \ldots, \xi_n)$

be an orthonormal set in $H_0^* = H_0$. $F$ determines a measure on the cylinder sets of $H_0$,

$$\tilde{\mu}(x: (\langle x, \xi_{n_1}\rangle, \ldots, \langle x, \xi_{n_k}\rangle) \in A_{n_1} \times \cdots \times A_{n_k}) \tag{2}$$
$$= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{A_{n_k}} \exp(-u^2/2)\, du.$$

We define another norm on $H_0$, $\langle x, x\rangle = \Sigma \alpha_n^2 \langle x, \xi_n\rangle_0^2$, which is weaker than $\langle\ ,\ \rangle_0$. Let $H_1$ be the completion of $H_0$ with respect to this norm and let $\{e_n\}$ be a complete orthonormal set in $H_1$. Then $\Sigma_n \langle x, e_n\rangle^2 = \Sigma\langle x, \alpha_n \xi_n\rangle_0^2$. We can define a measure on the cylinder sets of $H_1$ as follows:

$$\mu(x: (\langle x, e_{n_1}\rangle, \ldots, \langle x, e_{n_k}\rangle) \in B_{n_1} \times \cdots \times B_{n_k})$$
$$= \tilde{\mu}(x: (\langle x, \alpha_{n_1}\xi_{n_1}\rangle_0, \ldots, \langle x, \alpha_{n_k}\xi_{n_k}\rangle_0) \in B_{n_1} \times \cdots \times B_{n_k})$$
$$= \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi}\, \alpha_{n_i}} \int_{B_{n_i}} \exp(-u^2/2\alpha_{n_i}^2)\, du.$$

The last step is just (2). In other words a canonical normal distribution on $H_0$ induces a countably additive measure on $H_1$; the same measure induced by (1).

In the general theory we will consider Gaussian measures induced on a Banach space $B$ by a canonical normal distribution $F$ on a separable Hilbert space $H$. A norm or seminorm $\|x\|_1$ on $H$ is said to be measurable if for every $\varepsilon > 0$ there exists a finite dimensional projection $P_\varepsilon$ on $H$ such that $\text{Prob}(\|Px\|_1 > \varepsilon) < \varepsilon$ for all finite dimensional projections $P \perp P_\varepsilon$. (The measure which gives the probability is the measure induced by the canonical distribution on the cylinder sets of $H$. Note that $\{x: \|Px\|_1 > \varepsilon\} \subset H$.)

Given $H$, $F$ and a measurable norm or seminorm $\|\ \|_1$ let $B$ be the Banach space which is the completion of $H$ with respect to $\|\ \|_1$. Let $B^*$ be the dual of $B$; then $B^* \subset H^* = H \subset B$. The canonical distribution on $H$ induces a weak distribution on $B$ if $F$ is restricted to $B^*$. Let $\mu$ be the cylinder set measure on $B$ determined by this weak distribution. Gross' Theorem **[2]** states: Let $\|\ \|_1$ be a measurable norm on $H$ and $\mu$ the cylinder set measure on $B$ induced by the canonical distribution on $H$. Then $\mu$ extends to a countably additive Gaussian measure on $(B, \mathcal{B}(B))$, where $\mathcal{B}(B)$ is the $\sigma$-field of Borel sets in $B$.

Let $i$ denote the inclusion map of $H$ into $B$. The triple $(i, H, B)$ is called an abstract Wiener space. The measure induced on $B$ in Gross' Theorem is a Gaussian measure. Dudley, Feldman and Le Cam **[1]** show that if $\mu$ is a Gaussian measure in a real separable Banach space $B$ then (with an additional removable condition) there exists a real separable Hilbert space $H$ such that $(i, H, B)$ is an abstract Wiener space.

One obtains standard Brownian motion on [0, 1] in the following way. The Banach space is $C[0, 1]$ and the Hilbert space $H$ is the space of absolutely continuous functions $\{f(x): x \in [0, 1], f(0) = 0\}$ with inner product $\langle f, g\rangle = \int_0^1 f'g'$. Clearly $H \subset C[0, 1]$. The measurable norm on $H$ is the sup-norm on $C[0, 1]$. In other words $H$ is the reproducing kernel Hilbert space of the Brownian motion.

The reproducing kernel Hilbert space of a Gaussian process plays a critical role in abstract Wiener measure. Kallianpur [4] has shown the following: Let $C[0, 1]$ be the Banach space (with sup-norm $\| \|_1$) of all real valued continuous functions $[0, 1]$. Let $R$ be a continuous covariance on $[0, 1] \times [0, 1]$. Then the canonical normal distribution on $H(R)$ (the reproducing kernel Hilbert space determined by $R$) extends to a Gaussian probability measure on $\overline{H(R)}$, the closure of $H(R)$ in $C[0, 1]$, if and only if $\| \|_1$ is a measurable norm on $H(R)$. Therefore given a continuous Gaussian process on $[0, 1]$ the process can be realized as an abstract Wiener space $(i, H, B)$ where $H$ is the reproducing kernel Hilbert space determined by the covariance of the process.

The first half of Professor Kuo's book deals with abstract Wiener space. There are many steps in a proper development of this theory and in these lecture notes, prepared for a course at the University of Virginia, Professor Kuo goes through them carefully and presents them in a readable fashion. The first step is Gaussian measures on Hilbert space. Basic properties of trace class and Hilbert-Schmidt operators are given. These lead to the study of characteristic functionals on $H$ and a theorem of Prohorov which characterizes Gaussian measures on $H$ in terms of their characteristic functionals.

Abstract Wiener space is presented following Gross' original proof. This proof is more involved than Kallianpur's [4] and in a later section Kuo also gives the proof in [4]. The notes would be improved if the role of $H$ as a reproducing kernel Hilbert space was worked into the presentation. This section of the book is completed with a proof of the Gross-Sazonov Theorem.

There is a wealth of material in these one hundred pages. The presentation is careful and aside from some changes in notation it is easy to read. There are many examples and exercises to guide the reader, however it is hard to get an overview of the subject. These notes should have included some of the introductory comments of [2] and [4]. Also, I would have appreciated some remarks on the relation of this work to that of Dudley, Feldman and Le Cam [1].

There are other ways to define Borel measures on Banach spaces. The reader might wonder why workers have gone to the trouble of constructing abstract Wiener spaces. Gross explains this in the introduction of [2]. The work of Cameron and Martin (about 1945) on the equivalence of Wiener measure under certain types of translation and the computation of the relevant Radon-Nykodym derivatives seems to depend not on $C[0, 1]$, the space of paths of Brownian motion, but on the Hilbert space $H$ in the triple $(i, H, C[0, 1])$ that determines Brownian motion as an abstract Wiener space. The dependence was made apparent by the work of Segal [6], [7]. Abstract Wiener space is used by Kuelbs [5] to study the equivalence and singularity of Gaussian measures on any real separable Banach space. Kuo's notes follow this route. The next quarter of the book deals with the familiar theorems on the equivalence and singularity of Gaussian measures presented from the point of view of abstract Wiener spaces. The machinary developed leads to a simple proof of the equivalence of Wiener measure under translation and to the Feldman-Hajek Theorem on the equivalence and orthogonality of Gaussian measures on a real separable Hilbert space. Also, the Radon-Nykodym derivatives are easily computed. Kakutani's Theorem is stated without proof but Shepp's Theorem is proved. Results of Gross on the equivalence and orthogonality of abstract Wiener measures are also presented.

The last chapter begins with the result of Fernique (it is also due to Landau and Shepp) on the integrability of $\exp(\alpha\|X\|^2)$, for some $\alpha > 0$, for continuous Gaussian processes. ($\|\ \|$ indicates the sup-norm, integration is expectation on the measure space of the process.) Kuo also gives a proof of this result due to Skorohod. Skorohod's result appears weaker than Fernique's, but a simple argument (which was shown to me by S. R. S. Varadhan and is repeated in [3]) shows that they are equivalent. The idea behind Skorohod's proof is intriguing and should have other interesting applications.

The second part of the final chapter launches into analysis on abstract Wiener spaces. The role of Brownian motion in potential theory on $R^n$ has an analogue in the Wiener process with values in a Banach space and a potential theory on infinite dimensional spaces. Results of Gross on a potential theory on Hilbert space are presented. A generalized Laplacian can be defined on $H$ and a Dirichlet problem defined and solved. Similarly there is a theory of stochastic integrals and an analogue of Ito's Lemma. Thus, finally, Kuo gets to his own work and the work of other students of Gross. A more elaborate treatment of this work would be the next step if these Springer Notes are to be made into a book.

The stated purpose of the Lecture Notes in Mathematics is to quickly bring new material to a wide circle of readers. This is worthwhile and Kuo's notes are very useful. They are not a book. As a book the approach is too narrow. He fails to give sufficient attention to other works which have an important bearing on the subject. Also, at certain points, the reader is referred to a journal article in order to complete an argument. There is a widespread interest in the topics of these notes, in Gaussian processes and in probability limit theorems on Banach spaces. In each of these fields workers have their own ways of looking at things. It would be very useful to have a book that related these subjects, which, if it could not unify them, at least would clearly show what the interrelationships are.

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*Markov chains*, by D. Revuz, North-Holland Mathematical Library, vol. 11, North-Holland, Amsterdam; American Elsevier, New York, 1975, x + 336 pp., $35.50.

The theory of Markov processes can be considered in a great variety of settings. In the work under review the word "chain" is used to indicate discrete time (a convincing argument for this usage is made), the state space is a general measurable space, and the transition probabilities are assumed to be stationary. This context then determines the problems to be considered.

The most immediate problem, and historically the first to be pursued, concerns the asymptotic behavior of the $n$-step transition probabilities $P^n(x, A)$. In case the state space consists of a finite number of states only, this reduces to studying the asymptotic behavior of the $n$th power of a Markov matrix, and much early work was devoted to this situation. The case of general state space is of course much more complicated, and the pioneering work here is due to Doeblin. Between these two levels of generality lies that of denumerable state space, definitively treated by Kolmogorov, and alternatively by Feller.

In the ergodic theory of Markov chains one generally distinguishes between the recurrent and the transient case. Roughly, in the recurrent situation, a subset $A$ of the state space will be visited infinitely often by the Markov chain started at $x$, for all (or most) starting points $x$, provided only that $A$ is not too small (in a suitable sense). The parenthetical expressions can be made precise in various ways, leading to very different concepts of recurrence. In the denumerable case one may take "most" to mean all, and "small" to mean void. Following one of Doeblin's approaches for general state space, one can take "small" to mean of $\varphi$-measure zero, where $\varphi$ is an auxiliary measure on the state space. Then taking "most" to mean all, one obtains the notion of $\varphi$-recurrence. A chain that is $\varphi$-recurrent for some $\varphi$ is recurrent in the sense of Harris.

A subset $A$ of the state space is closed if $P(x, A) = 1$ for all $x \in A$. No matter what notion of recurrence is used, the first problem is to show that the state space can be broken up into minimal closed sets, and the Markov chain restricted to any one of these sets is either recurrent or transient.

In the recurrent case one hopes to establish that $P^n(x, \cdot)$ is asymptotically independent of $x$ (weak ergodicity); one can then expect convergence of