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THE BANACH SPACES $C(K)$ AND $L^p(\mu)$[1]

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My concern in this talk is with isomorphic, as opposed to isometric, properties of Banach spaces. I shall present here a limited discussion, from the isomorphic viewpoint, on the vast domain of the Banach spaces $C(K)$ and $L^p(\mu)$. The study of these spaces from this viewpoint leads to deep applications of many results in classical analysis and probability theory, and also to the discovery of new results which should be of classical interest. This study also provides a unified manner in which to comprehend a great deal of classical mathematics.

These special Banach spaces play a vital role in the study of general Banach spaces. They admit beautiful characterizations singling them out from the general theory. Their particular structure is rich and remarkable. Moreover, invariants for general Banach spaces have resulted from some of those initially established for these special spaces.

We take up the structure of quotient spaces and complemented subspaces of $C([0, 1])$ in §1. In §2 we briefly review the results concerning complemented subspaces of $L^p([0, 1])$ and in §3 we discuss reflexive subspaces of L^1. We also include some Banach-space consequences of the Radon-Nikodym property in an Appendix. We do not discuss here one of the powerful general techniques in Banach space theory, that of p-*summing* operators from a $C(K)$-space to a given Banach space. This technique was used in **[36]** for studying quotient spaces of $C(K)$-spaces; the methods developed there have turned out to hold in considerable generality. For an expository account of these developments, see **[39]**. We refer the reader to the recent book of Lindenstrauss and Tzafriri **[25]** for general background information on Banach space theory as well as much important information on such special spaces as the L^p and $C(K)$-spaces which we have not included here.

For the sake of convenience, we deal with real Banach spaces, although all stated results hold for complex ones as well. By a $C(K)$-space, we mean the space of all continuous real-valued functions on a compact Hausdorff space K, under the supremum norm; by an $L^p(\mu)$-space, the space of all equivalence classes of pth-power integrable functions defined on some measure space $(X, \mathcal{S}, \mu)$, under the norm $\|f\|_{L^p(\mu)}=(\int |f|^p\, d\mu)^{1/p}$. We use the notation $C=C(K)$

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for $K=[0, 1]$ and $L^p=L^p(\mu)$ for μ equal to Lebesgue-measure with respect to the Lebesgue measurable subsets of $[0, 1]$. In reality, C and L^p are the most important cases of $C(K)$ and $L^p(\mu)$-spaces; we shall primarily discuss only these, their sequential analogues c_0 and l^p, and their finite-dimensional analogues l_n^∞ and l_n^p. (For $1\leqq p\leqq\infty$, l^p (resp. l_n^p) equals $L^p(\mu)$ where μ is the "counting" measure with respect to the family of all subsets of N, the positive integers (resp. $\{1, 2, \cdots n\}$); c_0 denotes the space of all sequences vanishing at infinity under the supremum norm.)

For $\lambda\geqq 1$, Banach spaces X and Y are said to be λ-isomorphic if there exists an invertible linear operator T from X onto Y with $\|T\|\,\|T^{-1}\|\leqq\lambda$. If X is a subspace of Y, X is λ-complemented in Y if there is a linear projection P from Y onto X with $\|P\|\leqq\lambda$. The Banach spaces X and Y are said to be isomorphic (resp. isometric) if they are λ-isomorphic for some λ (resp. 1-isomorphic); if $X\subset Y$, X is said to be complemented in Y if it is λ-complemented in Y for some $\lambda\geqq 1$. By "operator" we mean a bounded linear operator; an operator $T:X\to Y$ is called an isomorphism if it is injective and bicontinuous between X and TX; it is not required that $Y=TX$; equivalently T is an isomorphism if there is a $\delta>0$ so that $\delta\,\|x\|\leqq\|Tx\|$ for all x in X.

Throughout, "Banach space" shall refer to an infinite-dimensional separable complete normed linear space, unless stated otherwise; "subspace" or "quotient-space" shall refer to a Banach subspace or quotient space of some Banach space.

1. **Complemented subspaces and quotient-spaces of C.** Why study C? We offer a few results as partial motivation. An early result of Mazur is that every Banach space is isometric to a subspace of C and a quotient space of L^1 (in fact of l^1). On the other hand, a striking result of Grothendieck asserts that a Banach space is isomorphic to Hilbert space if and only if it is isomorphic to a quotient of C and a subspace of L^1. (See **[11]**, **[23]**; derivations from the techniques presented here may be found in **[26]** and **[39]**.) We note incidentally that the isometric analogue of this result is open for all spaces of dimension n where $3\leqq n\leqq\infty$; however every 2-dimensional Banach space is isometric to a subspace of L^1 (cf. **[22]**). Thus the study of quotients of C is possibly a reasonable domain to try to comprehend; we shall see later that the structure of these spaces is quite rich.

Say that a Banach space B is universal if every Banach space is isomorphic to a subspace of B. A remarkable result of A. Pełczyński **[30]** asserts that a Banach space B is isomorphic to C if and only if B is universal and isomorphic to a complemented subspace of every other universal space. A theorem of Mulutin asserts that $C(K)$ is isomorphic to C for every uncountable compact metric space K (see **[27]** and **[5]**). Thus C is just a convenient representative of this isomorphism class. Actually, $C(\Delta)$ is often more convenient to work with than C, where Δ denotes the Cantor discontinuum. All of the results that we intend to discuss in some detail, with the exception of the Appendix, are consequences of the statements or techniques of proof of the following two more recent discoveries, due to the author.

THEOREM 1 [33]. *Every complemented subspace of C with nonseparable dual is isomorphic to C.*

THEOREM 2 [36]. *Every reflexive quotient space of C is isomorphic to a quotient space of L^p for some $2 \leq p < \infty$.*

Evidently Theorem 1 shows that Pełczyński's result may be sharpened as follows: A Banach space B is isomorphic to C if and only if B^* is nonseparable and B is isomorphic to a complemented subspace of every universal Banach space. It also provides a partial answer to the following fundamental open question: Is every complemented subspace of C isomorphic to $C(K)$ for some compact metric space K? The techniques of proof for all the known results related to this question involve a careful study of the dual of C, identified with the space of all Borel signed measures on $[0, 1]$, which of course can also be regarded as an $L^1(\upsilon)$-space. It may be that "honest" $C(K)$-techniques, rather than $L^1(\upsilon)$-techniques, need to be developed to answer this question.

We shall first summarize two of these other known results which bear on this problem, then pass to a sketch of the proof of Theorem 1. We first note the basic *weak compactness criterion*: *Let W be a bounded subset of C^*. If W is weakly compact, there exists a probability measure μ on* $[0, 1]$ *so that the members of W are uniformly absolutely continuous with respect to μ; i.e.* $\sup_{\nu \in W} |\upsilon(E)| \to 0$ *as $\mu(E) \to 0$. On the other hand, if W is not weakly compact, then there exist δ, ε with $0<\varepsilon<\delta$, disjoint open subsets $U_1, U_2, \cdots$ and $\mu_1, \mu_2, \cdots$ in W so that for all n, $|\mu_n(U_n)|>\delta$ and $|\mu_n|(\bigcup_{j\neq n} U_j)<\varepsilon$.* (The first assertion essentially follows from the Vitali-Hahn-Saks theorem (see [7]); the second assertion may be deduced from a result of Grothendieck (see [10] and [35]).) This criterion leads rather simply to

A. (A. Pełczyński [31]). *Every complemented subspace of C contains an isomorph of c_0.*

Let X be a complemented subspace of C and $T: C \to X$ a projection. The first part of the criterion yields Grothendieck's result that X cannot be reflexive, i.e. T cannot be weakly compact. For were T weakly compact, $L = T^* S_{X^*}$ would be weakly compact, where S_{X^*} denotes the unit ball of X^*. Now let $f_n \to 0$ weakly, (f_n) a sequence in C. Of course then $f_n \to 0$ pointwise. Choosing μ as in the criterion, choose (f'_n) a subsequence of (f_n) so that $f'_n \to 0$ μ-almost uniformly. It follows easily that $\|Tf'_n\| \to 0$; thus the unit ball of X would be compact, hence X would be finite-dimensional. (Of course the first part of the argument yields the Dunford-Pettis theorem (cf. [7, p. 494]): if T is a weakly compact operator on C, then $\|Tf_n\| \to 0$ for any sequence (f_n) with $f_n \to 0$ weakly.) Now applying the second part of the criterion, choose for each n a continuous φ_n of sup-norm one, supported in U_n, with $\int \varphi_n \, d\mu_n > \delta$, where ε, δ, (U_n), and (μ_n) are chosen as in the criterion. It then follows that setting $Y = [\varphi_n]$ (the closed linear span of the φ_n's in C), that Y is isometric to c_0 and $T|Y$ is an isomorphism. (Here too, we are really only using the fact that T is nonweakly compact to produce Y and hence the consequence that X contains an isomorph of c_0.)

B. (Lewis and Stegall [**21**]). Let X be a complemented *subspace of C and suppose X^* is separable. Then X^* is isomorphic to l^1.*

It is easily seen that X^* is isomorphic to a complemented subspace Y of $L^1[0, 1]$ which has the Radon-Nikodym property. Lewis and Stegall proved that any such space is isomorphic to l^1. Their result shows the paramount importance of this property to Banach space theory. We sketch a proof of their result and give some equivalences to the Radon-Nikodym property in the Appendix.

The proof of Theorem 1 requires the following special case, due to Pełczyński [**30**]: *A complemented subspace of C which contains an isomorph of C is itself isomorphic to C.* This is proved by applying the "decomposition method" together with an interesting topological result of Kuratowski's: if φ is a continuous map of a compact metric space K onto Δ, there is a subset U of K, homeomorphic to Δ, such that $\varphi|U$ is a homeomorphism. A deduction of this special case from Kuratowski's result, somewhat simpler than the one given in [**30**], may be found in [**12**, pp. 56–58]. Theorem 1 is an immediate consequence of this special case and the following theorem, which is in fact the main result of [**33**].

THEOREM 1′. *Let X be a (separable) Banach space and $T: C\to X$ an operator with T^*X^* nonseparable. Then there is a subspace Y of C with Y isometric to $C(\Delta)$ so that $T|Y$ is an isomorphism.*

Evidently Theorem 1′ implies that every quotient space of C with a nonseparable dual, contains an isomorph of C. One of the main themes of this talk is that the quotient spaces of C provide a special but extremely rich class of Banach spaces. An interesting result of Johnson and Zippin [**17**] asserts that this class includes all (separable) L^1-preduals, i.e. Banach spaces whose duals are isometric to $L^1(\mu)$ for some measure μ. It is immediate that if Z is a quotient space of C, then Theorem 1′ remains valid if "Z" is substituted for "C" in its statement.

The proof of Theorem 1′ is rather involved, using several classical techniques from other areas. We shall state the various ingredients of the proof, then show how they fit together. The basic approach is to set $W=\{T^*x^*:\|x^*\|\leqq 1, x^*\in X^*\}$, and then to prove that there is a Y as in the statement of Theorem 1′ so that W norms Y; i.e. so that for some $\delta>0$,

$$\delta\,\|y\|\leqq \sup_{w\in W}|w(y)| \quad \text{for all } y\in Y.$$

The argument actually shows that any nonseparable bounded W, contained in C^*, norms such a Y.

The ingredients consist of two reduction steps, a representation result, and an existence step.

STEP I. *The reduction to the case where W equals the unit ball of $l^1(\Gamma)$ for some uncountable set Γ.* By using transfinite induction, it is proved that W contains an uncountable family of almost-pairwise-singular measures. Precisely, *there is a $\delta>0$ such that for all $\varepsilon>0$, there exists an uncountable family $\{w_\alpha\}_{\alpha\in\Gamma}$ of* *elements of W and a family of pairwise singular signed Borel measures* $\{\mu_\alpha\}_{\alpha\in\Gamma}$ *such that for all* α, $\|\mu_\alpha\|\geqq\delta$ and $\|\mu_\alpha-w_\alpha\|\leqq\varepsilon$.

STEP II. *The reduction to the case where W equals the unit ball of a subspace Z isometric to* L^1. Precisely, it is proved that *there exists a subspace U of* C^*, *isometric and weak* isomorphic to* $C(\Delta)^*$, *such that for all* $f\in C$, $\sup|u(f)|\leqq\sup_\alpha \|\mu_\alpha\|^{-1}|\mu_\alpha(f)|$, *the supremum taken over all* $u\in U$ *of norm* 1. Of course, U contains a subspace Z isometric to L^1. This step is accomplished as follows: one first chooses a countable infinite subset of $\{\|\mu_\alpha\|^{-1}\mu_\alpha:\alpha\in\Gamma\}$ dense-in-itself in the weak*-topology. Then one constructs a subset L of the unit ball of C^*, homeomorphic to the Cantor set, so that the natural map j of C into $C(L)$ defined by $(jf)(l)=l(f)$ is actually a surjective quotient map. This construction involves modifications of arguments of Pełczyński **[29]** and Stegall **[41]**.

STEP III. The representation of a subspace Z of C^*, isometric to L^1, is as follows: *There is a Borel probability measure* μ *on* $[0, 1]$, *a Borel measurable function* θ *such that* $|\theta|\equiv 1$ *and a* σ*-algebra* $\mathscr{S}$ *of the Borel subsets of* $[0, 1]$ *so that* $\mu|\mathscr{S}$ *is a purely nonatomic measure and* $Z=\theta L^1(\mu|\mathscr{S})$; *i.e.* $Z=\{v\in C^*: dv=\theta f\,d\mu$ *where f is* $\mathscr{S}$*-measurable and* μ*-integrable*$\}$. (I am indebted to Professor T. Ito for pointing out that the statement of Proposition 2 of **[33]** is incorrect; the function θ as given in Step III was omitted in **[33]**. The proof of Step IV involves only a minor modification of the argument in **[33]**, using Lusin's theorem. Full details of the necessary corrections will appear.)

STEP IV. *The existence step, which easily produces the desired Y, consists of the following density lemma*: *Let* μ, θ *and* $\mathscr{S}$ *be as in Step* III. *For every* $\varepsilon>0$, *there is a continuous map* φ *from a compact subset K of* $[0, 1]$ *onto* Δ *with* $\theta\,|\,k$ *continuous relative to k, so that for every nonempty closed-and-open subset B of* Δ, *there is an* $S\in\mathscr{S}$ with $\varphi^{-1}(B)\subset S$ and $\mu(\varphi^{-1}(B))\geqq(1-\varepsilon)\mu(S)>0$. That is, there is a Cantor-like subset $\varphi^{-1}(K)$ so that the $\mathscr{S}$-measurable sets are ε-dense in the clopen subsets of $\varphi^{-1}(K)$. There is some delicacy in the construction of $\varphi^{-1}(K)$, which in appearance is like the classical construction of the Cantor set. The concept of conditional expectation is used in an essential way in this construction.

These ingredients are put together as follows: First choose δ as in Step I, then let $\varepsilon>0$ be such that $\delta-2\varepsilon\delta-\varepsilon>0$; let the μ_α's and w_α's be chosen as in Step I. Now choose U as in Step II, then select Z a subspace of U isometric to L^1.

Representing Z as in Step III, we have for all $h\in C$,

$$\sup\left|\int hf\,d\mu\right|\leqq\sup_{\alpha\in\Gamma}\left|\int h\,d\mu_\alpha/\|\mu_\alpha\|\right|,$$

the sup of the left side extending over all $f\in\theta L^1(\mu|\mathscr{S})$ with $\|f\|_1\leqq 1$. Now choose K and ϕ satisfying the conclusion of Step IV. Finally, let $E: C(K)\to C$ be an isometric extension operator; i.e. E is a linear isometry so that $Ef|K=f$ for all $f\in C(K)$ (the existence of such an E is a standard exercise in most real variable courses). Let $Y=E(\theta\varphi^{\#}(C(\Delta)))$ where $\varphi^{\#}h=h\circ\varphi$ for all $h\in C(\Delta)$. The proof that W norms Y is completed by using the above inequality and the conclusion of Step IV to show that $\sup_{\alpha\in\Gamma}|w_\alpha(h)|\geqq\delta-2\varepsilon\delta-\varepsilon$ for any norm-one function h which is the image under E of a finite-valued function in $\theta\varphi^{\#}(C(\Delta))$.

We conclude our discussion of Theorem 1 with the following open-ended question: Can one give nontrivial conditions on an operator $T:C\to C$ which imply that T^*C^* is nonseparable? For example, is this always the case if $T|X$ is an isomorphism for a reflexive X, or perhaps even for an X which contains no isomorph of c_0?

2. **Complemented subspaces of L^p.** One of the fascinating problems in this area is the determination of the isomorphic classification of complemented subspaces of L^p. Let $1<p<\infty$, $p\neq 2$. There are ten known distinct isomorphism types of complemented subspaces of L^p; it is an open question if there are infinitely many isomorphism types.[2] There is also quite a nice theory connected with these spaces; we briefly sketch some of the known results.

Let $1\leqq p\leqq\infty$. A Banach space is called an $\mathscr{L}_p$ space if there is a constant K so that each of its finite-dimensional subspaces is contained in another subspace which is K-isomorphic to a finite-dimensional L^p-space; i.e. to l^p_n for some n. These spaces were introduced in **[23]**, where it was proved that if B is an $\mathscr{L}_p$-space, then B^{**} is isomorphic to a complemented subspace of $L^p(\mu)$ for some μ; evidently for separable B and $1<p<\infty$, B itself is isomorphic to a complemented subspace of L^p.

In **[24]** it is proved that every complemented subspace of an L^p-space is either an $\mathscr{L}_p$-space or an $\mathscr{L}_2$-space. Moreover, it is shown that an $\mathscr{L}_p$-space has the stronger property that there is a K so that each of its finite-dimensional subspaces is contained in a K-complemented K-isomorph of a finite-dimensional L^p-space. For even "tighter" properties of $\mathscr{L}_p$-spaces, see **[32]**.

Now let $1<p<\infty$, $p\neq 2$. It can be seen that the closed linear span of a sequence of independent standard normal random variables is complemented in L^p and isometric to l^2. Thus the complemented subspaces of L^p are precisely the $\mathscr{L}_p$-spaces, up to isomorphism, and of course l^2. If for some λ, X_1, $X_2,\cdots$ are each λ-complemented subspaces of L^p, then so is $(X_1\oplus X_2\oplus\cdots)_p$ (which by definition equals all sequences (x_n) with $x_n\in X_n$ for all n and $\|(x_n)\|=(\sum\|x_n\|_p^p)^{1/p}<\infty$). Since l^2 is a complemented subspace of L^p, one easily obtains the five spaces L^p, l^p, l^2, $l^2\oplus l^p$, and $(l^2\oplus l^2\oplus\cdots)_p$ as factors of L^p. l^2 may also be realized as a complemented subspace of L^p by considering the span of a sequence of 2-valued symmetric identically distributed independent random variables. It is proved in **[37]** that if x_1, $x_2,\cdots$ is a sequence of 3-valued symmetric independent random variables, then $[x_j]_p$, the closed linear span of these variables in L^p, is again a complemented subspace of L^p. This space will be isomorphic either to l^2, l^p, $l^2\oplus l^p$ or a different space, denoted X_p. The latter occurs if, for example,

$$(\Delta)\qquad w_n\to 0 \quad\text{and}\quad \sum w_n^{2p/(p-2)}=\infty,$$

[2] ADDED IN PROOF. This has recently been answered in the affirmative by G. Schechtman.

where $p>2$ and $w_n=\|x_n\|_2/\|x_n\|_p$ for all n. It is shown in [**37**] that for any sequence of scalars (α_j), $\sum \alpha_j x_j$ converges in L^p if and only if

$$\max\{(\sum \alpha_n^2 w_n^2)^{1/2}, (\sum |\alpha_n|^p)^{1/p}\}<\infty.$$

X_p is then defined to be the space of all sequences of scalars (α_j) satisfying the above condition, where (w_n) is any fixed sequence of positive reals satisfying (Δ). X_p is thus realized as a subspace of $l^2 \oplus l^p$. Indeed, if (e_n) denotes the usual basis of l^2 and (b_n) the usual basis of l^p, then X_p equals the closed linear span in $l^2\oplus l^p$ of the sequence $(w_n e_n + b_n)$. It is proved that X_p is not a continuous linear image of $l^2 \oplus l^p$; for further properties, see [**38**].

Another space, B_p, is constructed as follows: For each n, let $B_{p,n}$ consist of all square summable sequences (α_j) of scalars under the norm

$$\|(\alpha_j)\|_{B_{p,n}} = \max\{n^{-(p-2)/2p}(\sum |\alpha_j|^2)^{1/2}, (\sum |\alpha_j|^p)^{1/p}\}.$$

One shows that there is a λ depending only on p so that $B_{p,n}$ is λ-isomorphic to a λ-complemented subspace of L^p, by considering a sequence of identically distributed 3-valued symmetric independent random variables $x_1, x_2, \cdots$ so that $\|x_1\|_2/\|x_1\|_p = n^{-(p-2)/2p}$. One then sets $B_p=(B_{p,1}\oplus B_{p,2}\oplus \cdots)_p$. Combining X_p and B_p with the spaces mentioned above, one finally obtains the five additional spaces X_p, B_p, $(l^2\oplus l^2 \oplus \cdots)_p \oplus X_p$, $B_p \oplus X_p$, and $(X_p \oplus X_p \oplus \cdots)_p$ as factors of L^p.

It is a rather important fact that L^p itself has an unconditional basis, the Haar-basis (cf. [**3**] and [**28**]). Evidently every subsequence of the Haar basis spans a complemented subspace of L^p. However Gamlen and Gaudet show in [**9**] that the possible isomorphism types thus obtained are only l^p and L^p. It is proved in [**15**] that every (separable) $\mathscr{L}_p$-space has a basis (for $p=1$ or ∞ also). It is unknown whether every $\mathscr{L}_p$-space has an unconditional basis. Schechtman proves in [**40**] that every $\mathscr{L}_p$-space with an unconditional basis is isomorphic to a complemented subspace of L^p spanned by a block basis of the Haar basis.

Results of Johnson, Zippin, and Odell show that the richness of the $\mathscr{L}_p$-space theory requires the presence of Hilbert space. Indeed, Johnson and Zippin prove in [**16**] that every $\mathscr{L}_p$-subspace of l^p is isomorphic to l^p; on the other hand, Johnson and Odell prove in [**14**] that for $p>2$, any subspace of L^p imbeds in l^p if it does not contain an isomorph of l^2. It follows that an $\mathscr{L}_p$-space which contains no isomorph of l^2 is isomorphic to l^p for $1<p<\infty$ (for $1<p<2$ one needs the additional fact that Hilbert space-isomorphs in L^p contain complemented-in-L^p subspaces (cf. [**32**]).

One of the deepest problems in this area is the characterization of the complemented subspaces of L^1. In contrast to the reflexive L^p-case, it seems likely that there are only two isomorphic possibilities; l^1 and L^1. We have previously noted the striking result of Lewis and Stegall that every complemented subspace of L^1 with the Radon-Nikodym property is isomorphic to

l^1. We also note the recent difficult result of Enflo; for $1 \leqq p < \infty$ if $X \oplus Y = L^p$, then either X or Y is isomorphic to L^p; that is, L^p is a primary Banach space.

In a different direction, one can consider the class of Banach spaces X such that *every* embedding of X into L^p is complemented. It is known that for $2 \leqq p < \infty$, this class consists of Hilbert space alone, while for $1 < p < \frac{4}{3}$, the class is empty. The determination of the class is an open question for $p = 1$ and $\frac{4}{3} \leqq p < 2$.

For $p > 2$, the fact that Hilbert space has this property is proved in [**18**]; a "localized version" is given in [**32**]. It is shown in [**37**] that l^p contains uncomplemented isomorphs of itself for all $2 < p < \infty$. Since every complemented subspace of L^p contains a complemented isomorph of l^p (provided it is not isomorphic to Hilbert space), the result follows for $p > 2$.

It does seem likely that the class is empty for $\frac{4}{3} \leqq p < 2$; however the situation for $p = 1$ seems unclear to me. The basic question here is: does l^1 contain an uncomplemented isomorph of l^1? For some equivalences related to this as well as references for the case $1 < p < \frac{4}{3}$, see [**24**].

3. **Reflexive subspaces of L^1 and a variation-of-density lemma.** The remainder of our discussion shall be concerned with Theorem 2 and the rather general techniques which arise from its proof. It is known that for all $2 \leqq p < \infty$, L^p is isometric to a quotient space of C; for $2 \leqq p < q < \infty$, L^p is isometric to a quotient space of L^q but l^q is not isomorphic to a quotient space of L^p. Thus Theorem 2 is "tight" in a sense.

All these results are established by duality, which immediately leads to the study of reflexive subspaces of L^1. The fact that L^r isometrically imbeds in L^s for all $1 \leqq s < r \leqq 2$ is a fairly simple consequence of the existence and properties of stable random variables of exponent r; moreover (for $r \neq 2$) there is no other known way of establishing this fact. We recall briefly the definitions; fix $1 < r \leqq 2$. A symmetric stable random variable of exponent r is by definition a measurable function f defined on some probability space $(X, \mathcal{S}, \mu)$ so that for some nonzero c, $\int_X e^{itf(x)}\, d\mu(x) = \exp(-|ct|^r)$ for all real t. For the sake of definiteness, we shall take X to be $[0, 1]$, $\mathcal{S}$ the Lebesgue measurable sets, μ Lebesgue measure.

It is a nontrivial fact (established in the '30's) that such a function f exists; moreover one has $\int_0^1 |f|^s\, dt < \infty$ for all $s < r$ (cf. most standard texts on probability theory or [**38**]). It follows from the method of characteristic functions that if $x_1, x_2, \cdots$ are independent identically distributed symmetric stable random variables of exponent r, then for any n and choice of scalars $c_1, \cdots, c_n$, $\sum c_j x_j$ has the same distribution (i.e. has the same nondecreasing rearrangement) as $(\sum |c_j|^r)^{1/r} x_1$. Consequently for any $s < r$,

$$\|\sum c_j x_j\|_s = (\sum |c_j|^r)^{1/r}\, \|x_1\|_s$$

which shows immediately that the closed linear span of the x_j's in L^s is isometric to l^r. (For isometrically imbedding L^r into L^s, see [**2**] and [**23**].)

Theorem 2 itself, by duality, follows immediately from the following result, which is the main point of [**36**]:

THEOREM 2′. *Let $1 \leqq p < 2$ and X be a subspace of L^p. Then either X contains a complemented isomorph of l^p or there is a $p' > p$ so that X imbeds in $L^{p'}$.*

(We restrict ourselves to L^p here only for convenience. Actually L^p could be replaced by $L^p(\mu)$ for any measure μ on some measurable space.)

Of course an immediate consequence is the somewhat surprising fact that every reflexive subspace of L^1 imbeds in L^p for some $p > 1$. The techniques used are those of the basic paper of Kadeć and Pełczyński [**18**] and a variation-of-density lemma to be discussed shortly.

If $p=1$ and X is not reflexive or $p>1$ and the p and 1-norms are not equivalent on X, then the techniques of [**18**] yield that for all $\lambda > 1$, X contains a λ-complemented (in L^p) λ-isomorph of l^p. On the other hand, if X is reflexive in case $p=1$ or if the p and 1-norms on X are equivalent for the case $p>1$, it is proved that there is a probability density φ on $[0, 1]$ (i.e. $\int_0^1 \varphi\, dt = 1$ and $\varphi > 0$) and a $p' > p$ such that $\int_0^1 |f/\varphi|^{p'} \varphi\, dt < \infty$ for all $f \in X$. This means that X is imbedded in $L^{p'}$ in a surprisingly simple fashion.

In turn the existence of this density is demonstrated as follows: If there is no such φ and $p' > p$, then one may use the theory of p-absolutely summing operators to show that for each $p' > p$ there exist finite dimensional-subspaces X' of X so that $\inf_\varphi \sup_{x'} (\int |x'/\varphi|^{p'} \varphi\, dt)^{1/p'}$ is arbitrarily large (where the infimum is taken over all probability densities φ, the supremum over all x' in X' of L^1-norm one). Alternatively if one is only interested in the existence of an imbedding, one may simply note that if this could not be done then all finite-dimensional subspaces of X would uniformly imbed in L^p for some $p' > p$, which would yield that X itself imbeds (cf. [**23**]). The variation-of-density lemma then shows that for each $\varepsilon > 0$ and n, there exist n norm-1 elements in X which "$1+\varepsilon$-dominate" the usual l_n^p-basis, in the L^1-norm. Properties of stable random variables are finally used to show that the p and 1-norms on X could not have been equivalent after all (or in the $p=1$ case, one obtains that the unit ball of X is not uniformly integrable and consequently X is not reflexive). Rather than using properties of stable random variables, one may use instead a truncation lemma due to P. Enflo and the author [**8**]. A detailed exposition of this alternate route is given in [**39**].

We pass now to a discussion of this lemma, which may be formulated as follows:

VARIATION-OF-DENSITY LEMMA. *Let X be a Banach space, $(\Omega, \mathcal{S}, \mu)$ a probability space, $1 < p < \infty$, and $T: X \to L^1(\mu)$ a norm-1 operator such that*

$$N = \sup_{\|x\| \leqq 1} \|Tx\|_{L^p(\mu)} < \infty \tag{1}$$

and

$$N \leqq \sup_{\|x\| \leqq 1} \|Tx/\varphi\|_{L^p(\varphi\, d\mu)} \tag{2}$$

for all positive μ-probability densities φ.

Then given n and $0 < \delta < 1$, there is an M depending only on p, n, and δ, so that if $N \geqq M$, there exist $x_1, \cdots, x_n$ in X of norm one and disjoint measurable sets $E_1, \cdots, E_n$ so that $N\delta \leqq (\int_{E_i} |Tx_i|^p\, d\mu)^{1/p}$ for all i.

This result is proved in [**36**] for the case where $X \subset L^1(\mu)$ and T is the identity injection; this case is all that is needed for Theorem 2. As observed by Maurey [**26**], exactly the same proof as in [**36**], namely that of the Sublemma on p. 362, yields the more general statement. We prefer, however, to phrase the argument somewhat differently than in [**36**] so that the intuitions involved will be more apparent. (For expositions of consequences of this more general statement of the lemma, see [**39**] and [**26**].)

The result is proved by playing off the following two facts against each other. The first fact is standard: given f in $L^p(\mu)$ with $\|f\|_p$ almost N (which is assumed very large) and $\|f\|_1 \leqq 1$, there must exist a set F of small measure so that $(\int_F |f|^p \, d\mu)^{1/p}$ is also almost N. Precisely, let $\bar{\delta} < 1$ and suppose F is a measurable set such that $\int_F |f|^p \, d\mu \geqq \bar{\delta}^p N^p$ and $\|f\|_1 \leqq 1$. Then setting $E = \{x : |f(x)| > N\} \cap F$, we have

$$(3) \qquad \mu(E) < \frac{1}{N} \quad \text{and} \quad \int_E |f|^p d\mu \geqq \bar{\delta}^p N^p - N^{p-1}.$$

The second fact is proved by the variation-of-densities technique, and is as follows: Given E of small measure, there exists an $x \in X$ of norm one so that $\int_{\sim E} |Tx|^p \, d\mu$ is almost N^p. We see this by applying (2) to a probability density φ of the form $\varphi = a\chi_E + b\chi_{\sim E}$ where a and b are constants. Assume $\mu(E) \leqq n/N$. The requirement $\int \varphi \, d\mu = 1$ means that $1 = a\mu(E) + b(1 - \mu(E))$, or

$$b = \frac{1 - a\mu(E)}{1 - \mu(E)} \geqq 1 - a\mu(E) \geqq 1 - \frac{an}{N}.$$

Now given $\eta > 0$, we can choose x with $\|x\| = 1$ and $\int |Tx|^p / \varphi^{p-1} \, d\mu > N^p - \eta$, by (2). Evaluating,

$$\begin{aligned} \int |Tx|^p/\varphi^{p-1} \, d\mu &= \frac{1}{a^{p-1}} \int_E |Tx|^p \, d\mu + \frac{1}{b^{p-1}} \int_{\sim E} |Tx|^p \, d\mu \\ &\leqq \frac{N^p}{a^{p-1}} + \frac{1}{b^{p-1}} \int_{\sim E} |Tx|^p \, d\mu \end{aligned}$$

by (1), hence

$$(4) \qquad \int_{\sim E} |Tx|^p \, d\mu \geqq \left(1 - \frac{an}{N}\right)^{p-1} \left[N^p \left(1 - \frac{1}{a^{p-1}}\right) - \eta \right].$$

To achieve what we wish, we let a depend on N in such a way that $a(N) \to \infty$ yet $a = o(N)$. A convenient choice is to simply let $a = N^{1/2}$.

Now intuitively, the argument goes as follows. By (1), choose x_1 of norm one so that $\|Tx_1\|_p$ is almost N. Choose E_1 by the first fact so that $(\int_{E_1} |Tx_1|^p \, d\mu)^{1/p}$ is also almost N, E_1 of small measure. Now by the second fact, choose x_2 of norm one so that $(\int_{\sim E_1} |Tx_2|^p \, d\mu)^{1/p}$ is almost N. By the first fact, choose $E_2 \subset \sim E_1$, E_2 of small measure, so that $(\int_{\sim E_2} |Tx_2|^p \, d\mu)^{1/p}$ is almost N. Since $E_1 \cup E_2$ will be of small measure, by the second fact we may choose x_3 of norm 1 so that $(\int_{\sim(E_1 \cup E_2)} |Tx_3|^p \, d\mu)^{1/p}$ is almost N. As we have set things up, we may continue this process n times (so that the conclusion of the lemma holds) if we first choose $\delta < \bar{\delta} < 1$, then choose M so large that for all $N>M$,

$$(1-n/\sqrt{N})^{p-1}N^p(1-N^{-(p-1)/2})>\bar{\delta}^pN^p>\bar{\delta}^pN^p-N^{p-1}>\delta^pN^p.$$

Since η can be then arbitrarily small, for each $j<n$, setting $E=\bigcup_{i=1}^j E_i$, we will have $\mu(E)\leqq j/N\leqq n/N$. Consequently the $x=x_{j+1}$ that we choose satisfying (4) will have the property that $\int_{\sim E}|Tx|^p\,d\mu\geqq\bar{\delta}^pN^p$; hence by (3) we can choose $E_{j+1}\subset\sim E$ with $\mu(E_{j+1})<1/N$ and $\int_{E_{j+1}}|Tx|^p\,d\mu>\delta^pN^p$.

Maurey observed in **[26]** that this lemma leads to results of considerable generality; his formulations also clarify the L^p-case. Let $1\leqq q<p\leqq 2$, let X be an arbitrary Banach space, and $T:X\to L^q$ a given operator. Suppose there is no K and no probability density φ on $[0, 1]$ so that

$$\left(\int |Tx/\varphi^{1/q}|^p\,\varphi\,dt\right)^{1/p}\leqq K\,\|x\|$$

for all $x\in X$. Then Maurey's generalization asserts that for all n and $\lambda>1$, there exist an operator S from X to l_n^p with $\|S\|\leqq\lambda$ and norm-one elements $x_1,\cdots,x_n$ in X so that $Sx_i=e_i$ for all i, where $(e_1,\cdots,e_n)$ is the usual basis for l_n^p.

For the case $q=1$, one may use the theory of p-absolutely summing operators to show that for all $M<\infty$, there is a μ and a T satisfying the hypotheses of the variation-of-density lemma for some $N>M$. The functions $(Tx_i)_{i=1}^n$ in its conclusion are almost-disjointly supported, hence for δ close enough to 1, a standard perturbation argument produces a good projection from L^p onto their linear span which will in fact be almost isometric to l_n^p. This projection in turn leads to the operator S. Precisely, the standard perturbation argument yields that λ and p given, there is a $\delta<1$ so that if $f_1,\cdots,f_n$ are elements of the unit ball of $L^p(\mu)$ for which there exist disjoint measurable sets $E_1,\cdots,E_n$ with $\delta\leqq(\int_{E_i}|f_i|^p\,d\mu)^{1/p}$ for all i, then there exists an operator $U:L^p\to l_n^p$ with $\|U\|\leqq\lambda$ and $Uf_i=e_i$ for all i (i.e. the f_i's span a space almost isometric to l_n^p which is the range of an almost-contractive projection). Thus one only has to choose M large enough so that the conclusion of the lemma is satisfied for n and δ; then $S=(1/N)UT$ is the desired operator. The general case requires a minimax lemma due to Maurey, although its finite-dimensional version does not. Again, one reduces to the lemma; the proof holds with almost no changes if one simply replaces "$L^1(\mu)$" by "$L^q(\mu)$" everywhere in its statement. Maurey **[26]** has also used these techniques to obtain results about subspaces of L^p for $p<1$. As mentioned before, a detailed exposition of the consequences of the lemma to general Banach space theory, in terms of the notion of p-absolutely summing operators, is given in **[39]**.

We pass now to some consequences of Theorem 2′ and also some open questions. Let μ be a probability measure on some measurable space, let X be a subspace of $L^1(\mu)$, and let $1<p<\infty$. Let $I_p(X)$ equal the infimum of those numbers K so that there is a μ-probability density φ with

$$\left(\int |x|^p(t)\varphi^{1-p}(t)\,d\mu(t)\right)^{1/p}\leqq K\int|x(t)|\,d\mu(t)\quad\text{for all } x\in X;$$

if there is no such K, put $I_p(X)=\infty$. It follows from the results of [36] that for $1<p\leqq 2$, $I_p(x)<\infty$ if and only if X is isomorphic to a subspace Y of $L^p(\nu)$ for some probability measure ν, such that the p and 1 norms are equivalent on Y. By our preceding remarks, we thus obtain that if $I_p(X)<\infty$ for some $p<2$, then $I_{p'}(X)<\infty$ for some $p'>p$; that is, the set of $p<2$ such that $I_p(X)<\infty$, forms an open interval. Moreover, the interval is nonempty provided X is reflexive; suppose this is the case. Let q be the right-end point of this interval. Does X contain a subspace which imbeds in L^q? We have been able to show that an affirmative answer implies an affirmative answer to the question: Does X contain a subspace isomorphic to l^r for some r, $1<r\leqq 2$?

As shown in [36], $I_p(X)$ is actually attained by a particular choice of a probability density φ (with the property that for all $x\in X$, the null set of x is contained up to a set of measure zero in the null set of φ); moreover, $I_p(X)=\sup I_p(F)$, the supremum taken over all finite-dimensional subspaces F of X. This shows the finite-dimensional character of these concepts and results.

It would seem appropriate to find natural classes of spaces X so that

(*) X is already contained in L^p if $I_p(X)$ is finite.

One such class has been found by Bachelis and Ebenstein [1]; namely, the class of subspaces X of $L^1(\mu)$, where μ is normalized Haar-measure on a compact abelian group G and X is a translation invariant subspace. A slightly different version of their argument is as follows: Let $K=I_p(X)<\infty$. Choose a probability density φ on G so that for all x in X,

$$\int_G |x|^p(g)\varphi^{1-p}(g)\,d\mu(g)\leqq K^p\,\|x\|_1^p.$$

Now fix x in X; then for each g' in G, since X is translation invariant,

$$\int |x(g+g')|^p\,\varphi^{1-p}(g)\,d\mu(g)\leqq K^p\,\|x\|_1^p.$$

Integrating this inequality with respect to g' and changing the order of integration, one obtains that

$$\int |x(g)|^p\,d\mu(g)\int \varphi^{1-p}(g)\,d\mu(g)\leqq K^p\,\|x\|_1^p.$$

If we assume that X is nonzero, then the translation invariance of X implies that φ is strictly positive almost everywhere. Hence

$$\int \varphi^{1-p}(g)\,d\mu(g)\geqq \left(\int\left(\frac{1}{\varphi}\right)\varphi\,d\mu\right)^p=1;$$

therefore $\|x\|_p\leqq K\,\|x\|_1$ for all $x\in X$. (Actually, this argument holds for arbitrary compact groups G, not just the abelian ones.)

It is a standard fact that such an X equals the closed linear span in L^1 of a subset E of the characters of G. E is called a $\Lambda(p)$-set if the q- and p-norms are equivalent on X for all $0<q<p$. As shown in [1], the above considerations

yield that for all $1 \leqq p < 2$, every $\Lambda(p)$-set is a $\Lambda(p')$-set for some $p' > p$. It is an open question if every such set is already a $\Lambda(2)$-set. (For other applications of these ideas to harmonic analysis, see pp. 83–86 of **[26]**.)

Another class of spaces satisfying (*) are those subspaces X of L^1 equal to the closed linear span of a sequence $x_1, x_2, \cdots$ of identically distributed independent random variables. This fact is easily deduced from the results of **[36]** as follows: Suppose $I_p(X) < \infty$. Then Lemma 3 of **[36]** yields that $\int |x_1|^p \, dt < \infty$. Now let y be in X, and let $N_1, N_2, \cdots$ be infinite disjoint subsets of the positive integers. We may choose for each j a y_j in $[x_n]_{n \in N_j}$ so that y_j has the same distribution as y. But then $y_1, y_2, \cdots$ is also a sequence of identically distributed independent random variables and $I_p([y_j]) \leqq I_p(X) < \infty$. Hence $\int |y_1|^p \, dt = \int |y|^p \, dt < \infty$, i.e. $X \subset L^p$.

We conclude with the following open question concerning a permanence property of subspaces of L^p. Suppose $1 < p \leqq 2$ and X is a Banach space such that $(X \oplus X \oplus \cdots)_p$ imbeds in L^1. Does X imbed in L^p? By suitably varying the techniques so far mentioned we have been able to show (unpublished) that under these hypotheses, X does imbed in L^q for all $q < p$. It follows for all $1 \leqq q < p$, that $(l^q \oplus l^q \oplus \cdots)_p$ does not imbed in L^1.

Appendix. Complemented subspaces of L^1 and the Radon-Nikodym property. In **[21]**, Lewis and Stegall prove the following remarkable result:

THEOREM A1. *Let B be an infinite-dimensional complemented subspace of $L^1([0, 1])$, and suppose that B is isomorphic to a conjugate Banach space. Then B is isomorphic to l^1.*

We present here alternate derivations of this and some of the other structural results obtained by Lewis and Stegall. These results make use of the notion of a Radon-Nikodym derivative. The main tool is Theorem A3, which yields the fact that a Banach space has the Radon-Nikodym property if and only if every operator from L^1 into the space factors through l^1. (All results presented are known, some implicit in the writings of the above authors and earlier ones such as A. Grothendieck.)

Throughout, we let $(\Omega, \mathcal{S}, \mu)$ denote a probability space; λ shall denote a (not-necessarily-σ-finite) measure on some measurable space, and B an infinite-dimensional Banach space. We recall that a function $\varphi : \Omega \to B$ is called *simple* if it is measurable with finite range; it is called *strongly measurable* if there exists a sequence (φ_n) of simple functions from Ω to B with φ_n tending to φ pointwise almost everywhere. We let $\|\cdot\|_\infty$ denote the essential supremum norm on bounded B-valued measurable functions. The following result is easily established, using Egoroff's theorem:

PROPOSITION A2. *Let $\varphi : \Omega \to B$ be a bounded strongly measurable function and $\varepsilon > 0$. Then there exist disjoint measurable subsets $\Omega_1, \Omega_2, \cdots$ of Ω with $\Omega = \bigcup_{n=1}^\infty \Omega_n$ and simple B-valued functions $(\varphi_{nj})_{n,j=1}^\infty$ so that for each n, φ_{nj} is supported on Ω_n for all j, $\sum_j \|\varphi_{nj}\|_\infty < \|\varphi\|_\infty + \varepsilon$, and $\sum_j \varphi_{nj}(\omega) = \varphi(\omega)$ for almost all $\omega \in \Omega_n$.*

An operator $T:L^1(\mu)\to B$ is said to be differentiable if there exists a bounded strongly measurable function $\varphi:\Omega\to B$ so that $Tf=\int_\Omega \varphi(\omega)f(\omega)\,d\mu(\omega)$ for all $f\in L^1(\mu)$.

(An important theorem of Dunford, Pettis and Phillips asserts that every weakly compact operator T is differentiable.) We note that if T and φ satisfy this relationship, then φ is uniquely determined up to a change on a set of measure zero, and moreover

$$\|T\|=\|\varphi\|_\infty. \tag{5}$$

We call φ the derivative of T. (To distinguish these notions from the entirely different ones of differentiation in the calculus sense, perhaps the terminology "Radon-Nikodym-differentiable" and "Radon-Nikodym-derivative" would be more suitable.)

It is trivial that if S and T are differentiable operators from $L^1(\mu)$ to B with derivatives φ and ψ, respectively, then for any scalars a and b, $aS+bT$ is differentiable with derivative $a\varphi+b\psi$. Thus the differentiable operators from $L^1(\mu)$ to B form a Banach space (in the operator norm) isometric to the space of all equivalence classes of bounded strongly measurable functions from Ω to B, under the essential supremum norm.

We note in passing that operators of the form $T:L^1(\mu)\to B$ are in one-to-one correspondence with B-valued measures ν of bounded variation λ. Indeed, given T, one sets $\nu(E)=T(\chi_E)$ for all $E\in\mathcal{S}$. On the other hand, given ν and λ, one may put $\mu(E)=\lambda(E)/\lambda(\Omega)$ for all $E\in\mathcal{S}$; it is then easily seen that there is a $T:L^1(\mu)\to B$ with $\|T\|=\lambda(\Omega)$ so that $\nu(E)=T(\chi_E)$ for all $E\in\mathcal{S}$.

B is said to have the *Radon-Nikodym property* (or *R.N. property*) if for every μ on a measurable space, every $T:L^1(\mu)\to B$ is differentiable. It is known that one may restrict oneself to operators defined on $L^1([0,1])$. The connection between operators and B-valued measures yields that B has the R.N. property if and only if for every μ on a measurable space and every B-valued measure ν with μ-continuous bounded variation λ, there exists a Bochner-μ-integrable function φ so that $\nu(E)=\int_E \varphi\,d\mu$ for all $E\in\mathcal{S}$. Of course this φ will not be bounded in general; however the scalar-valued Radon-Nikodym theorem gives a positive μ-integrable g and a bounded ψ so that $\varphi=\psi g$ a.e.

It is easily seen that l^1 has the R.N. property. More generally, any subspace of a separable dual has the R.N. property. For this and other related results see the expository paper **[4]** and the references given there.

The following factorization theorem is our main tool in obtaining structural results such as A1 from Radon-Nikodym derivatives. Its proof is fairly self-contained, resting only on A2 and the fact that l^1 has the R.N. property.

THEOREM A3. *An operator $T:L^1(\mu)\to B$ is differentiable if and only if it can be factored through l^1; i.e. there exist operators $U:L^1(\mu)\to l^1$ and $V:l^1\to B$ so that $T=VU$. When this occurs, then for each $\varepsilon>0$, the operators U and V may be chosen so that $\|U\|\leqq\|T\|+\varepsilon$ and $\|V\|=1$.*

PROOF. Suppose first that T admits such a factorization. Then U is differentiable; let ψ be the derivative of U. Then $V\circ\psi$ is easily seen to be the derivative of T.

Now suppose T is differentiable with derivative φ. We are indebted to D. Lewis for the observation that T may be realized as an "l^1-sum" of compact operators, each of which may be factored through l^1 by the "lifting property" of $L^1(\lambda)$-spaces. A direct elementary proof is as follows: we consider three cases in increasing order of complexity.

Case 1. φ is simple. We may then choose n and disjoint measurable sets $E_1, E_2, \cdots, E_n$, each of positive measure, and $b_1, \cdots, b_n$ in B so that $\varphi=\sum_{i=1}^n b_i\chi_{E_i}$.

Let $e_1, e_2, \cdots$ be the usual basis for l^1; define $U:L^1(\mu)\to l^1$ by $Uf=\|T\|\sum_{j=1}^n(\int_{E_j} f\,d\mu)e_j$ for all $f\in L^1(\mu)$ and $V:l^1\to B$ by $V(e_j)=\|T\|^{-1}b_j$ for $1\leqq j\leqq n$ and $V(e_j)=0$ for $j>n$. Since $\|T\|=\sup_i\|b_i\|$, we have that $\|V\|=1$; evidently $\|U\|=\|T\|$ and $T=VU$.

Case 2. There are $\delta>0$ and simple functions $\varphi_1, \varphi_2, \cdots$ so that $\sum\varphi_j=\varphi$ and $\sum\|\varphi_j\|_\infty<\|\varphi\|_\infty+\delta$. We shall show that then U and V may be chosen with $T=VU$ and $\|V\|=1$, $\|U\|<\|\varphi\|_\infty+\delta$.

For each j, define T_j by $T_jf=\int\varphi_jf\,d\mu$ for all $f\in L^1(\mu)$; then choose $U_j:L^1(\mu)\to l^1$ and $V_j:l^1\to B$ so that $T_j=V_jU_j$, $\|V_j\|=1$, and $\|U_j\|=\|T_j\|$ (which of course equals $\|\varphi_j\|_\infty$). Now $\sum\varphi_j$ must converge to φ in the $\|\cdot\|_\infty$-norm and $\|\sum_{j=1}^n T_j-T\|=\|\sum_{j=1}^n\varphi_j-\varphi\|_\infty$ for all n, by (5); hence $\sum_{j=1}^\infty T_j=T$. Let $X=(l^1\oplus l^1\oplus\cdots)_1$; of course X is isometric to l^1. Define $U:L^1(\mu)\to X$ by $Uf=(U_jf)_{j=1}^\infty$ for all $f\in L^1(\mu)$; define $V:X\to B$ by $V((b_j))=\sum V_jb_j$ if $(b_j)\in X$ and $b_j\in l^1$ for all j. Then $\|U\|\leqq\sum\|U_j\|=\sum\|T_j\|=\sum\|\varphi_j\|_\infty<\|\varphi\|_\infty+\delta$, $\|V\|=1$, and $T=VU$.

Case 3. This is the general case. By Proposition A2, we may choose $\Omega_1, \Omega_2, \cdots$ and $(\varphi_{nj})_{n,j=1}^\infty$ as in its statement. Fix j and let μ_j be the measure on $(\Omega_j, \mathscr{S}\cap\Omega_j)$ obtained by restricting μ. The argument for Cases 1 and 2 did not require that μ be a probability measure; hence we may choose $U_j:L^1(\mu_j)\to l^1$ and $V_j:l^1\to B$ so that $\|V_j\|=1$, $\|U_j\|<\|T\|+\varepsilon$ and $V_jU_j=T|L^1(\mu_j)$. Let X be as in Case 2 and define $U:L^1(\mu)\to X$ and $V:X\to B$ by $Uf=(U_nf\cdot\chi_{\Omega_n})_{n=1}^\infty$ and $V((b_n))=\sum V_nb_n$. Fixing $f\in L^1(\mu)$, we have that

$$\|Uf\|\leqq\sum_{n=1}^\infty\|U_n\|\,\|f\cdot\chi_{\Omega_n}\|_1\leqq(\|T\|+\varepsilon)\sum\|f\cdot\chi_{\Omega_n}\|_1\leqq(\|T\|+\varepsilon)\,\|f\|.$$

Moreover $VUf=Tf$ and $\|V\|=1$, so the theorem is proved.

Theorem A1 may now be proved as follows: Let B satisfy its hypotheses and let $P:L^1\to B$ be a bounded linear projection from L^1 onto B. Since separable duals have the R.N. property, B does also; hence by A3, there exist operators $U:L^1\to l^1$ and $V:l^1\to B$ with $P=VU$. Since $VU|_B=I|_B$, it follows that $U|B$ is an isomorphism between B and the subspace $U(B)$ of l^1; then UV is a projection from l^1 into $U(B)$, so $U(B)$ is isomorphic to l^1 by a result of Pełczyński **[31]**. (Of course this argument shows immediately that any complemented subspace of L^1 with the R.N. property is isomorphic to l^1. We also note that a result in **[34]** asserts that a conjugate space isomorphic to a complemented subspace of $L^1(\mu)$ is separable and hence isomorphic to a complemented subspace of $L^1([0, 1])$. Thus Theorem A1 holds if "$L^1([0, 1])$" is replaced by "$L^1(\mu)$ for some probability measure μ" in its statement.)

We wish finally to indicate some results in a nonseparable setting. Let Γ be a set; $l^1(\Gamma)$ denotes the space of all scalar-valued functions f defined on Γ with $\|f\|_1=\sum_{\gamma\in\Gamma}|f(\gamma)|<\infty$. Given Γ and Banach spaces $(E_\alpha)_{\alpha\in\Gamma}$, $(\bigoplus_{\alpha\in\Gamma} E_\alpha)_1$ denotes the space of all $\bigcup_{\alpha\in\Gamma} E_\alpha$-valued functions f defined on Γ with

$$\|(f_\alpha)_{\alpha\in\Gamma}\|_1 = \sum_{\alpha\in\Gamma} \|f_\alpha\|_{E_\alpha} < \infty.$$

The following factorization theorem is a simple consequence of A3.

PROPOSITION A4. *A Banach space B has the R.N. property if and only if for every $L^1(\lambda)$-space and operator $T:L^1(\lambda)\to B$, there exist a set Γ and operators $U:L^1(\lambda)\to l^1(\Gamma)$ and $V:l^1(\Gamma)\to B$ so that $T=VU$.*

PROOF. Since $l^1(\Gamma)$-spaces have the R.N. property, the "if" part is immediate. Suppose B has the R.N. property and $T:L^1(\lambda)\to B$ is a given operator. If λ is σ-finite, then $L^1(\lambda)$ is isometric to $L^1(\mu)$ for some μ, whence A3 gives the result. Assume that λ is not σ-finite. Then using Zorn's lemma, we may choose an uncountable set Γ and probability measures μ_α on certain measurable spaces for all $\alpha\in\Gamma$, so that $L^1(\lambda)$ is isometric to $(\bigoplus_{\alpha\in\Gamma} L^1(\mu_\alpha))_1$. For notational convenience let us identify these two spaces. Now fix $\alpha\in\Gamma$. Let $T_\alpha:L^1(\mu_\alpha)\to B$ be the obvious operator induced by T; by A3, we may choose $U_\alpha:L^1(\mu_\alpha)\to l^1$ and $V_\alpha:l^1\to B$ with $T_\alpha=V_\alpha U_\alpha$, $\|V_\alpha\|=1$, and $\|U_\alpha\|\leqq 2\|T_\alpha\|$. Let $E_\alpha=l^1$; then evidently $(\sum_{\alpha\in\Gamma}\oplus E_\alpha)_1$ is isometric to $l^1(\Gamma)$.

Now simply define $U:L^1(\lambda)\to(\bigoplus_{\alpha\in\Gamma} E_\alpha)_1$ and $V:(\bigoplus_{\alpha\in\Gamma} E_\alpha)_1\to B$ by $Uf=(U_\alpha f)_{\alpha\in\Gamma}$ and $V((f_\alpha)_{\alpha\in\Gamma})=\sum V_\alpha(f_\alpha)$.

The final structural result that we give has an important special case: If a $\mathscr{L}_1$-subspace of $L^1([0, 1])$ is isomorphic to a subspace of a separable dual, then it is isomorphic to a subspace of l^1. (See §2 for the definition of $\mathscr{L}_1$-spaces.)

THEOREM A5 (LEWIS AND STEGALL [**21**]). *If the Banach space B is isomorphic to a complemented subspace of $L^1(\lambda)$ for some λ and B has the R.N. property, then B is isomorphic to $l^1(\Gamma)$ for some set Γ. If B is an $\mathscr{L}_1$-space which is isomorphic to a subspace of a conjugate Banach space with the R.N. property, then B is isomorphic to a subspace of $l^1(\Gamma)$ for some set Γ.*

REMARK. Let μ be given. If Γ is an uncountable set, then $l^1(\Gamma)$ does not imbed in $L^1(\mu)$ (cf. [**34**, p. 214]). Consequently if B is a complemented subspace of $L^1(\mu)$ with the R.N. property, then B is isomorphic to l^1, while if B is a $\mathscr{L}_1$-subspace of $L^1(\mu)$ which imbeds in a dual space with the R.N. property, then B imbeds in l^1.

PROOF OF A5. The first assertion follows from A4 and the result that a complemented subspace of $l^1(\Gamma)$ is isomorphic to $l^1(\Gamma')$ for some Γ' (due to Köthe [**19**]; cf. also [**34**]) in the same way that A1 follows from A3 and Pełczyński's result.

It is known that $\mathcal{L}_1$-spaces are isomorphic to spaces whose second conjugate spaces are isometric to complemented subspaces of $L^1(\lambda)$-spaces. We may then assume without loss of generality that there is a λ so that $B^{**} \subset L^1(\lambda)$ and so that there is a projection $P: L^1(\lambda) \to B^{**}$. We may also choose a conjugate Banach space X with the R.N. property and an isomorphic imbedding $T: B \to X$ (where TB is a *subspace* of X). Now there is a projection Q from X^{**} onto X (regarded as canonically imbedded in X^{**}). The operator $\tilde{T}$ defined by $\tilde{T} = QT^{**}P$ may then be factored through $l^1(\Gamma)$ for some set Γ by operators U and V as in Proposition A4. Since $\tilde{T}|B = T$ is an isomorphism, it also follows that $U|B$ is an isomorphism, proving the theorem.

REMARK. The proof shows that A5 remains valid if one replaces "isomorphic to a subspace of a conjugate Banach space with the R.N. property" by the formally weaker statement "isomorphic to a subspace of a space X with the R.N. property such that X is complemented in X^{**}," in its statement.

A fundamental open question is the following: Is every complemented subspace of L^1 isomorphic to l^1 or L^1? The work of Lewis and Stegall shows that this is equivalent to the question: If a complemented subspace of L^1 fails to have the R.N. property, is it isomorphic to L^1? By some recent (unpublished) work of Enflo, this question has an affirmative answer if the following one does: If a subspace of L^1 fails to have the R.N. property, does it contain a subspace isomorphic to L^1?

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