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ON CUSPIDAL REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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ABSTRACT. Let k be a p-adic field, and G a reductive connected algebraic group over k. Fix a maximal torus T of G which splits in an unramified extension of k, and which has the same split rank as the center of G. For each character θ of $T(k)$, satisfying some conditions, there is a cuspidal representation γ_θ of $G(k)$ which is a sum of a finite number of irreducible representations; the correspondence $\theta \mapsto \gamma_\theta$ is one-to-one on the orbits of such characters by the little Weyl group of T; furthermore, the formulas for the formal degree of γ_θ and its character for sufficiently regular elements of $T(k)$ are given: they are formally the same as is the discrete series for real reductive groups.

1. **Unramified maximal tori.** Let k be a p-adic field, that is a finite extension of $\mathbf{Q}_p$ or a field of formal series over a finite extension of $\mathbf{F}_p$. We denote by $\bar{k}$ the residue field of order q.

Let G be a reductive connected algebraic group defined over k, the derived group G_{der} of which is simply connected. A maximal torus of G defined over k is called *minisotropic* if it normalizes no (proper) horocyclic subgroup of G defined over k.

LEMMA. *Suppose there exists a minisotropic maximal torus T of G which splits in a finite unramified extension L of G. Then the Galois group Γ of L over k has a unique fixed point v in the apartment of T in the building of $G_{\mathrm{der}}(L)$* [2]; *moreover, the face of v is minimal amongst the faces in this apartment which are invariant by* Γ.

2. **Characters.** We conserve notations and hypotheses of §1 and the Lemma. Let θ be a continuous character of $T(k)$. For each $\lambda \in X^\vee(T)$, the lattice of rational one-parameter subgroups of T, we define a character θ_λ of $L^\times$ by

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$$\theta_\lambda(z) = \theta\left(\prod_\Gamma{}^\gamma(z^\lambda)\right), \qquad z \in L^x.$$

DEFINITION 1. The character θ is called *regular* if, for every root α of (G, T), the character θ_{α^υ} of L^x associated to the coroot α^υ is nontrivial.

For each root α, we denote by $|\alpha|_\theta$ the conductor of θ_{α^υ} and let R_f be the set of roots of (G, T) such that $|\alpha|_\theta \leqslant f$.

DEFINITION 2. The character θ is called *good* if it is regular and if, for every f, the set R_f is a convex set of roots.

We need the following form of *Macdonald's conjecture* (cf. [1, C-6.7]):

Let $\overline{S}$ be a reductive connected algebraic group over $\overline{k}$, $\overline{T}$ a minisotropic maximal torus of $\overline{S}$; fix a finite extension $\overline{L}$ of $\overline{k}$ which splits $\overline{T}$; let Γ be its Galois group. A character $\overline{\theta}$ of $\overline{T}(\overline{k})$ is called regular if, for every root α of $(\overline{S}, \overline{T})$, the character $z \longmapsto \overline{\theta}(\Pi_\Gamma^\gamma(z^{\alpha^\upsilon}))$ of $\overline{L}^x$ is nontrivial; if $\overline{\theta}$ is a regular character of $\overline{T}(\overline{k})$, then there exists a unique class $\overline{\sigma}_{\overline{\theta}}$ of representations of $\overline{S}(\overline{k})$ such that, if $\mathrm{St}_{\overline{S}}$ denotes the Steinberg representation of $\overline{S}(\overline{k})$, $\mathrm{St}_{\overline{S}} \otimes \overline{\sigma}_{\overline{\theta}}$ is the induced representation $\mathrm{Ind}(\overline{S}(\overline{k}), \overline{T}(\overline{k}), \overline{\theta})$. Moreover $\overline{\sigma}_{\overline{\theta}}$ is cuspidal, and the intertwining number of two such representations $\overline{\sigma}_{\overline{\theta}_1}$ and $\overline{\sigma}_{\overline{\theta}_2}$ is equal to the number of elements in the little Weyl group of $\overline{T}$ in $\overline{G}(\overline{k})$ which send $\overline{\theta}_1$ on $\overline{\theta}_2$.

3. Cuspidal representations.

THEOREM. *Let G be a reductive connected algebraic group over the p-adic field k, the derived group G_{der} of which is simply connected. Let T be a minisotropic maximal torus which splits in an unramified extension of k. Fix a good character θ of $T(k)$. Assume Macdonalds's conjecture and one of the following conditions*:

(i) $|\alpha|_\theta > 1$ *for every root α of $(G(L), T(L))$*

(ii) *the residual characteristic of k is not 2 and there exists a rational representation ρ of G_{der} such that the corresponding bilinear form* $\mathrm{Tr}\, \rho(X)\rho(Y)$ *on the Lie algebra* Lie $\overline{T}_{\mathrm{der}}(\overline{k})$ *is nondegenerate.*

Then there exists a class γ_θ of representations of $G(k)$ such that:

(a) γ_θ *is a finite sum of irreducible representations of $G(k)$, the coefficients of which have compact support modulo the center*;

(b) *the intertwining number of two such representations γ_{θ_1} and γ_{θ_2} is the number of elements in the little Weyl group $W(T)$ of T in $G(k)$ which send θ_1 on θ_2*;

(c) *there exists a Haar measure on* $G(k)$, *independent of* θ, *such that the formal degree of* γ_θ *is*

$$d(\theta) = \left(\prod_R q^{|\alpha|_\theta - 1}\right)^{1/2}$$

where R *is the set of roots of* (G, T);

(d) *for* $t \in T(k)$ *such that* $\operatorname{val}(t^\alpha - 1) \geqslant |\alpha|_\theta/3$ *for every* $\alpha \in R$, *the value on* t *of the character of* γ_θ *is given by*

$$\operatorname{Tr} \gamma_\theta(t) = (-1)^{l(G)}(-1)^{\Sigma_{R/\Gamma}(|\alpha|_\theta - 1)} \sum_{W(T)} \frac{\theta}{\Delta}({}^w t)$$

where $l(G)$ *is the split semisimple rank of* G, Δ *is the* $W(T)$*-invariant function on the regular elements of* $T(k)$ *given by*

$$\Delta(t) = (-1)^{\Sigma_{R/\Gamma} \operatorname{val}(t^\alpha - 1)} |\operatorname{Det}_{\operatorname{Lie} G/\operatorname{Lie} T}(\operatorname{Ad} t - 1)|^{1/2},$$

and Γ *is the Galois group over* $\bar{k}$ *of an unramified extension which splits* T.

4. **Remarks.** 1. The proof is based upon an explicit construction of γ_θ (assuming Macdonald's conjecture), obtained by inducing a finite dimensional representation of a compact open subgroup of $G(k)$ naturally associated to T and θ; the essential tool is given by Weil's paper about Heisenberg groups [5]; we used too an argument given by R. Howe [4].

2. In the case where $|\alpha|_\theta$ is constant and strictly greater than 1, and if the point v of the lemma is special, the proofs are given in [3].

3. G. Lusztig has just proved Macdonald's conjecture.

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