

A FUNCTIONAL CHARACTERIZATION OF MARKOVIAN LINEAR EXAVES

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Let A and B be compact Hausdorff topological spaces and π be a continuous function from A into B . As usual, $C(A)$ denotes the Banach algebra of all complex-valued and continuous functions over A . A linear operator $[\pi]$ from $C(B)$ into $C(A)$ is then canonically associated with π in such a way that for any f in $C(B)$ we get $[\pi]f = f \circ \pi$. According to [2] (see also [3]), a linear exave (for π) is a linear and bounded operator P from $C(A)$ into $C(B)$ such that the following diagram is commutative

$$\begin{array}{ccc} C(B) & \xrightarrow{[\pi]} & C(A) \\ [\pi] \downarrow & & \uparrow [\pi] \\ C(A) & \xrightarrow{P} & C(B) \end{array}$$

The purpose of this paper is two fold: to obtain a characteristic functional equation for linear exaves and to describe how such operators are mixing linear extension properties (EX) with averaging ones (AVE). So has been coined the word.

Some definitions will shorten the theorems to come. Let X be a compact Hausdorff topological space and let Y be a closed subset of X . The restriction operator $R: C(X) \rightarrow C(Y)$ associated with Y ($Rf(y) = f(y)$ for all y in Y) is a markovian operator (i.e. linear operator such that $\|P\| = P(1) = 1$). Tietze's theorem asserts that R is onto.

A linear extension operator for Y in $C(X)$ is a bounded and linear operator $E: C(Y) \rightarrow C(X)$ such that $R \circ E$ is the identity operator on $C(Y)$. Let \sim be a closed equivalence relation on X and let us denote by $\tilde{C}(X)$ the

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subalgebra of $C(X)$ containing all functions which are constant on the equivalence classes defined with \sim .

A linear q -extension operator, for Y and \sim , is a bounded and linear operator $Q: \tilde{C}(Y) \rightarrow C(X)$ such that $R \circ Q$ is the identity operator on $\tilde{C}(Y)$. From now on, we shall identify $\tilde{C}(X)$ and $C(X/\sim)$. Let us introduce some functional equations. It is convenient to say that a linear and continuous operator $P: C(X) \rightarrow C(X)$ which satisfies for all f, g in $C(X)$, $P(f \cdot Pg) = P(Pf \cdot g)$, is a multiplicatively symmetric operator. With the functional equation $P(f \cdot Pg) = Pf \cdot Pg$ we define averaging operators (or semimultiplicatively symmetric operator). Finally $P(f \cdot Pg) = P(f \cdot g)$ defines interpolation operators.

For any operator P in $C(X)$ the relation \mathfrak{P} on X , defined according to $x\mathfrak{P}y$ if $Pf(x) = Pf(y)$ for all f in $C(X)$, is a closed equivalence relation. It is the main tool we may associate with an averaging operator P . Concerning an interpolation operator, the main tool is a subset $K(P)$ defined as the intersection of the zero sets for all functions which belong to the kernel of P .

PROPOSITION 1 (see [5] or [10]). *There exists an interpolation operator on $C(X)$ having a closed subset Y as its $K(P)$ if and only if there exists a linear extension operator for Y in $C(X)$.*

Averaging operators on $C(X)$ have been characterized in [4]. We then get

THEOREM 1. *Let $P: C(X) \rightarrow C(X)$ be a markovian operator. The operator P is multiplicatively symmetric if and only if $P = Q \circ S \circ R$ where R is the restriction operator associated with a certain closed subset Y of X .*

S is an averaging and markovian operator on $C(Y)$ for which $S(C(Y)) = R(P(C(X)))$ and Q is a markovian linear q -extension operator for Y and for the equivalence relation \mathfrak{P} .

If we restrict ourselves to the algebra $C_R(X)$ of all real-valued functions in $C(X)$ we get

THEOREM 2. *Let $P: C_R(X) \rightarrow C_R(X)$ be a linear and bounded operator. Suppose that there exists a bounded sequence K_n in $C_R(X)$ such that $PK_n(x)$ converges pointwise towards 1 for all x in X . Then P is a multiplicatively symmetric operator if and only if $P = Q \circ S \circ R$ with the same definitions as in Theorem 1 except that S and Q need no longer be markovian operators.*

The set Y defined within Theorem 1 may be called the averaging set for

P . It is in fact the subset of all points y in X such that $P(f \cdot Pg)(y) = Pf(y) \cdot Pg(y)$. The problem is to prove first $Y \neq \emptyset$

Let P be a continuous operator on $C(X)$. We define A to be the averaging set of P and B to be the topological quotient space X/\mathfrak{P} . We denote the quotient mapping by $\pi: X \rightarrow X/\mathfrak{P}$ and $\tilde{\pi}$ its restriction to A . Then, A and B are compact subspaces. When A is not empty, we may define $\tilde{P}: C(A) \rightarrow C(B)$ according to $\tilde{P}g(\pi(x)) = P(\tilde{g})(x)$ where \tilde{g} is any given extension of g into an element of $C(X)$. Generally \tilde{P} depends upon \tilde{g} . However, we see that it is not the case for the following situation:

THEOREM 3. *If P is a markovian multiplicatively symmetric operator, then \tilde{P} is a markovian exave for $\tilde{\pi}$.*

Conversely, let A and B be two compact Hausdorff topological spaces. Let $\tilde{\pi}: A \rightarrow B$ be a continuous function and denote by $\tilde{\pi}(A)$ the image of A through $\tilde{\pi}$. Let $\pi: X \rightarrow B$ be any compact fiber bundle over B extending the given fiber bundle $\tilde{\pi}: A \rightarrow \tilde{\pi}(A)$. By R , we shall denote the restriction operator defined on $C(X)$ and associated with A ($R: C(X) \rightarrow C(A)$).

THEOREM 4. *If \tilde{P} is a markovian linear exave for $\tilde{\pi}$, then the linear operator $P: C(X) \rightarrow C(X)$, defined by $Pf(x) = \tilde{P}(Rf)(\pi(x))$ for all x in X , is a markovian linear multiplicatively symmetric operator.*

We may use the now classical notion of Choquet boundary (see [8] or [1] for nonseparating subspaces of $C(X)$).

PROPOSITION 2. *Using the same hypothesis and notations as in Theorem 1, we may replace operator Q occurring there by a markovian linear extension operator E if and only if the Choquet boundary of the range of operator P contains Y .*

We also get a new rephrasing (see [6] or [5]).

PROPOSITION 3. *Suppose that P is a norm-one idempotent and linear operator on $C(X)$, where X is a compact Hausdorff topological space. Suppose that the following property holds for the range of P :*

For two given real numbers $\lambda \neq \lambda'$, a relation like $e^{i\lambda}Pf(x) = e^{i\lambda'}Pf(x')$, for all f in $C(X)$, implies that x does not belong to the Choquet boundary of the range of P .

Then P is a multiplicatively symmetric operator on $C(X)$.

This last proposition has to be used in combination with Theorem 1 or Theorem 2.

Averaging operators, interpolation operators or multiplicatively symmetric operators have been defined with the help of some functional equations. We then have solved those functional equations on $L(C(X))$, algebra of all linear bounded operators on $C(X)$ where X is a compact Hausdorff topological space. More generally we have solved these equations on $L(A)$, where A is any commutative C^* -algebra (i.e. for example on $L(L^\infty(\Omega, F, \mu))$ where (Ω, F, μ) denotes a measure space). The general solution of such functional equations, but on (abelian) groups, is given in [9].

The proof of Theorem 1 will appear in [7].

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