BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, May 1975

H^p SPACES AND EXIT TIMES OF BROWNIAN MOTION[1]

BY D. L. BURKHOLDER

Communicated by Harry Kesten, January 14, 1975

Let R be a region of the complex plane, Z a complex Brownian motion starting at a point in R, and τ the first time Z leaves R:

$$\tau(\omega) = \inf\{t > 0: Z_t(\omega) \notin R\}.$$

There are several ways to study such exit times. Here we describe a new approach that gives rather precise information about the moments of τ.

We shall always assume for simplicity that R contains the origin and Z starts there: $Z_0(\omega) = 0$, $\omega \in \Omega$, where (Ω, A, P) is the underlying probability space. If F is a function analytic in the open unit disc D, let

$$\|F\|_{H^p} = \sup_{0<r<1} \left[\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta\right]^{1/p}$$

and

$$\|\tau^{1/2}\|_p = (E\tau^{p/2})^{1/p}.$$

THEOREM 1. *Suppose R is the range of a function F analytic and univalent in D with $F(0) = 0$. Then, for $0 < p < \infty$,*

$$c_p\|\tau^{1/2}\|_p \leqslant \|F\|_{H^p} \leqslant C_p\|\tau^{1/2}\|_p. \tag{1}$$

In particular,

$$\tau^{1/2} \in L^p(\Omega, A, P) \Leftrightarrow F \in H^p(D).$$

If R is simply connected and has a nondegenerate boundary, then such a function F exists by the Riemann mapping theorem.

In (1), as elsewhere in this note, the choice of the positive real numbers c_p and C_p depends only on p.

The right-hand side of (1) is true in a more general setting. Let Φ be a continuous nondecreasing function on $[0, \infty]$ with $\Phi(0) = 0$ and $\Phi(2\lambda) \leqslant \gamma\Phi(\lambda)$, $\lambda > 0$.

AMS (MOS) subject classifications (1970). Primary 60J65, 30A78, 31A05, 60J45.
[1]Work partially supported by NSF.

THEOREM 2. *Suppose F: $D \longrightarrow R$ is analytic (but not necessarily univalent) in D with $F(0) = 0$. Then*

$$\sup_{0<r<1} \int_0^{2\pi} \Phi(|F(re^{i\theta})|)\, d\theta \leqslant c_\gamma E\Phi(\tau^{1/2}).$$

The left-hand side of (1) also holds more generally:

THEOREM 3. *Suppose F is a function analytic in D with $F(0) = 0$, and, for almost all θ, the nontangential limit of F at $e^{i\theta}$ exists and belongs to the complement of R. Then the left-hand side of* (1) *is satisfied.*

Here is a simple application of Theorem 1. Fix $0 < \alpha \leqslant 2$ and let R be the set of all complex numbers $re^{i\theta} - 1$ satisfying $r > 0$ and $|\theta| < \alpha\pi/2$. Then R and F, defined by $F(z) = ((1 + z)/(1 - z))^\alpha - 1$, satisfy the conditions of Theorem 1 and an easy calculation gives $\tau^{1/2} \in L^p \Leftrightarrow p < \alpha^{-1}$. Note the Brownian motion Z does not take much longer to hit $(-\infty, -1]$, the complement of R corresponding to $\alpha = 2$, than it does to hit the larger parabolically shaped complement of the range of $(1 + z)^{-2} - 1$, $z \in D$. In both cases, $\tau^{1/2} \in L^p$ for $p < ½$ and $\tau^{1/2} \notin L^p$ for $p \geqslant ½$.

In general, if R is simply connected and has a nondegenerate boundary, then $\tau^{1/2} \in L^p$ for $p < ½$. This follows from Theorem 1 and the classical result [2, p. 50] that a function F analytic and univalent in D satisfies $F \in H^p$, $0 < p < ½$. A related statement, the proof of which rests partly on recent results of Baernstein [1], is

THEOREM 4. *Let R_s be the region obtained from R by circular symmetrization and let τ_s be the corresponding exit time. Then*

$$\|\tau^{1/2}\|_p \leqslant c_p \|\tau_s^{1/2}\|_p, \qquad 0 < p < \infty.$$

The next result is closely related to Theorem 1 but does not require that R be simply connected.

THEOREM 5. *If $0 < p < \infty$, then $\tau^{1/2} \in L^p$ if and only if there is a function u harmonic in R such that $|z|^p \leqslant u(z)$, $z \in R$. If u is the minimal harmonic function satisfying this inequality, then*

$$c_p \|\tau^{1/2}\|_p \leqslant [u(0)]^{1/p} \leqslant C_p \|\tau^{1/2}\|_p.$$

Hansen [3] defined $h(R)$, the Hardy number of the region R, to be the supremum of all $p \geqslant 0$ such that $|z|^p$ is majorized by a harmonic function in R. Let

$$e(R) = \sup\{p \geq 0: E\tau^{p/2} < \infty\}.$$

This might be called the exit number of the region R. By Theorem 5, $e(R) = h(R)$. This has diverse applications. For example, by Theorem 4, $e(R_s) \leq e(R)$; therefore $h(R_s) \leq h(R)$.

Theorem 5 holds also in $\mathbf{R}^n$.

Further results, applications, and proofs will appear elsewhere.

REFERENCES

1. Albert Baernstein, *Integral means, univalent functions and circular symmetrization,* Acta Math. 133 (1974), 139–169.

2. Peter L. Duren, *Theory of H^p spaces,* Pure and Appl. Math., Vol. 38, Academic Press, New York and London, 1970. MR 42 #3552.

3. Lowell J. Hansen, *Hardy classes and ranges of functions,* Michigan Math. J. 17 (1970), 235–248. MR 41 #7118.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801