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PROPER T-MAPS OF T-MODULES

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In investigating homotopy equivalences of smooth G-manifolds where G is a compact Lie group, Petrie [3], [4], [5] makes use of proper G-maps of degree 1 from one G-module to another of the same complex dimension. The first nontrivial example of such a map, called a quasi-equivalence, was given by Petrie [6] for two-dimensional S^1-modules. Necessary and sufficient conditions for the existence of a quasi-equivalence when G is any compact Lie group are now known [2]. For simplicity, the case where G is a torus T is outlined here.

Definitions and notation. Let M and N be T-modules of the same dimension, and let $\hat{T}$ be the group of irreducible T-modules. If there is a T-module Q such that a quasi-equivalence $\omega\colon N + Q \to M + Q$ exists we say that there is a stable quasi-equivalence of N to M, and we write $N \leqslant M$. Let $P = (p_1, \cdots, p_u)$ be a set of pairwise relatively prime integers with $u \geqslant 2$, or $P = (-1)$, and let $\mathbf{P} = \Pi(\psi^{p_i} - 1)$ be the associated Adams operation in $R(T)$, the complex representation ring of T. Petrie has conjectured this

THEOREM. *$N \leqslant M$ iff there are nonnegative integers $a_{\mathbf{P},\chi}$ such that $M - N = \Sigma_{\chi\in\hat{T}}\ \Sigma_{\mathbf{P}}\ a_{\mathbf{P},\chi}\mathbf{P}\chi$, in $R(T)$.*

The proof [1] uses K_T-theory for the necessity in an argument suggested by Petrie. The sufficiency is shown by constructing the required maps.

Let any T-module X be written as $X^T + X_T$, where X^T is the set fixed pointwise by the action of T. If $N \leqslant M$, then the fact that $\omega\colon N + Q \to M + Q$ is equivariant and proper leads, by an argument using a commutative diagram in K_T-theory, to

(i) $\dim M^T = \dim N^T$, and

(ii) $r(t) = (\lambda_{-1}(M_T))/(\lambda_{-1}(N_T)) \in R(T)$, where $t = (t_1, \cdots, t_n) \in T$ represents the indeterminates in the expression for $R(T)$ as a Laurent polynomial ring over the integers.

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The fact that ω has degree 1 requires

(iii) $|r(1)| = 1$.

Now, each irreducible T-module in M_T or N_T contributes a factor of the form $(1 - t^x)$ to $r(t)$, where $t^x = t_1^{x(1)} \cdots t_n^{x(n)}$ describes the action of $t \in T$, with the $x^{(i)}$ integers not all zero. These factors are partitioned into classes in the jth of which all x's are multiples of a common n-vector x_j. Then the factors in that class are expressed as products of cyclotomic polynomials in the indeterminate t^{x_j}. Considerations of reducibility and the fact that $r(t) \in R(T)$ require that all such cyclotomic polynomials in the denominator of $r(t)$ also appear in the numerator. After cancellation of such factors, what remains is a product of cyclotomic polynomials,

$$r(t) = \prod_j \prod_k \phi_{m_{j,k}}(t^{x_j}).$$

The fact that $\deg \omega = 1$ requires that each $m_{j,k}$ not be a power of a prime. Then each $\phi_{m_{j,k}}$ can be written as a ratio of factors of the form $(1 - (t^{x_j})^d)$ with an equal number of factors in the numerator and denominator. Here the d's are positive integers determined by $m_{j,k}$. If we write T-modules

$$M_{j,k} = \sum_{d\ (\text{numerator})} (t^{x_j})^d \quad \text{and} \quad N_{j,k} = \sum_{d\ (\text{denominator})} (t^{x_j})^d,$$

it is true that

$$M_{j,k} - N_{j,k} = \prod_h (\psi^{p_h} - 1)(t^{x_j})^p,$$

where $m_{j,k} = p\ \Pi_h\ p_h$ with the p_h all the distinct prime factors. Then $\Sigma_{j,k}(M_{j,k} - N_{j,k})$ is of the form given in the Theorem, and it is also equal to $M - N$.

For sufficiency, we observe that $M - N$ is expressed as the sum of terms of the form $\mathbf{P}\chi$ each of which can be thought of as some $M_{\mathbf{P},\chi} - N_{\mathbf{P},\chi}$. We construct $\omega_{\mathbf{P},\chi}$: $N_{\mathbf{P},\chi} \to M_{\mathbf{P},\chi}$ with the required properties, and take the direct sum of the maps, which is ω: $N + Q \to M + Q$. For this construction, beginning with Petrie's two-dimensional example for S^1, which corresponds to $u = 2$ in $P = (p_1, \cdots, p_u)$, we devise a 2^{u-1}-dimensional quasi-equivalence and prove by induction on u that it has the required properties [1]. The same map is good when $\chi \in \hat{T}$ too. The maps turn out to be polynomials in the complex variables and their conjugates, with normalizing adjustments and smoothing factors. Also, the above outline of a proof has assumed that in the irreducible T-modules t^x each x is a positive multiple of its x_j. The case where some are negative multiples can be treated by modifying the maps slightly.

The Theorem gives conditions for stable quasi-equivalences, but actual quasi-equivalences exist under virtually the same conditions [2].

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