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THE DETERMINANT OF A RANDOM MATRIX

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An experimental worker measures $n \times n$ independent magnitudes a_{ij} $(i, j = 1, \cdots, n)$ and computes the determinant of the corresponding matrix before discarding the a_{ij}'s. One can imagine a set-up where he only gets to know the determinant but not the a_{ij} themselves. By repeating the same experiment over and over and averaging the corresponding determinants he obtains the determinant of the unknown mean values m_{ij} of the magnitudes a_{ij}.

The purpose of this note is to indicate that under reasonable assumptions he can get much more information about the *unknown matrix* m_{ij} *at no extra cost.*

Assume that the entries a_{ij} $(i, j = 1, \cdots, n)$ are jointly Gaussian variables with *unknown* means m_{ij} and *known* correlation matrix R.

For the remainder of this note we concentrate on the special case where the entries are independent and have the same variance, i.e. $R = \rho I$ $(\rho \neq 0)$; this case already well illustrates our point. The results given here illustrate the fact that in many nonlinear identification problems the presence of noise can prove helpful (see [1]). We give first a complete analysis of the 2×2 case.

For the discussion below put

$$M(x) = \begin{pmatrix} a + x_1 & b + x_2 \\ c + x_3 & d + x_4 \end{pmatrix}$$

with *a, b, c, d unknown* constants, and x_1, x_2, x_3, x_4 Gaussian (0, 1) random variables. The case $(0, \rho)$ $(\rho \neq 0)$, is just the same.

Bring in the characteristic function of the determinant $F(\lambda) = Ee^{i\lambda \det M}$ and conclude after some elementary computation that

$$F(\lambda) = e^{i\lambda M(0)}\Psi(a + d, -\lambda)\Psi(d - a, \lambda)\Psi(b + c, -\lambda)\Psi(b - c, \lambda).$$

Here

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$$\Psi(\alpha, \lambda) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-i\lambda)\xi^2} e^{i\lambda\alpha\xi} \frac{d\xi}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}(1-i\lambda)^{-1}\lambda^2\alpha^2}}{(1-i\lambda)^{\frac{1}{2}}}$$

so that one finally gets

$$F(\lambda) = \frac{e^{i\lambda \det M(0)}}{(1-i\lambda)^2} \times e^{-\lambda^2} \frac{(a^2+b^2+c^2+d^2) + i\lambda 2(ad-bc)}{(1-i\lambda)(1+i\lambda)}.$$

In conclusion, we see that the knowledge of the distribution function of $\det M(x)$ is equivalent to that of $\det M(0) = ad - bc$, and $\operatorname{tr} MM^* = a^2 + b^2 + c^2 + d^2$. Moreover it is plain that this information is already contained in the first two moments of $\det M(x)$. This is summed up in

THEOREM I. *From the distribution function of* $\det M(x)$ *one obtains exactly* $\det M(0)$ *and* $\operatorname{tr} M(0)M^*(0)$.

COROLLARY. *If the matrix of mean values* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is symmetric, then from the distribution of* $\det M(x)$ *one can deduce both eigenvalues of* $M(0)$ *up to a common change in sign.*

REMARK. If a_{12} and a_{21} had zero variances, and only a_{11}, a_{22} were subject to a random error of the same nonzero variance, our proof shows that a_{11} and a_{22} could be determined up to order and a common change in sign. Moreover $\det M(0)$ is known, and thus we also have the product $a_{12}a_{21}$.

Notice that we have found the determinant and the trace of $M(0)M^*(0)$, besides $\det M(0)$. It is in this form that the results above extend most naturally to higher dimensions. We have

THEOREM II. *The distribution function of* $\det M(x)$ *determines exactly the eigenvalues of* $M(0)M^*(0)$*—alias the singular values of* $M(0)$*—and* $\det M(0)$.

The proofs are not so simple as for $n = 2$. There is no closed form for the characteristic function of $\det M(x)$ and one has to compute moments and then solve the resulting system of equations. Some indications of how this is done will be given below.

If each entry has (known) *variance* ρ one shows that

$$E \det M(x) = \det M(0) \equiv \det M,$$

$$E(\det M(x))^2 = (\det M)^2 + \rho\Sigma(M^{ij})^2 + 2\rho^2\Sigma(M^{ijkl})^2 + \cdots + n!\rho^n.$$

On the last line the first sum runs over the square of all the $(n-1) \times (n-1)$ minors of M, the second one over all the $(n-2) \times (n-2)$ minors (squared) of M, and so on. This formula is all that is needed if the variance ρ is variable;

see below. The expressions for the higher moments involve these same quantities. The task is then to solve the appropriate system of equations. For instance in the case of $n = 3$ we have

$$E \det M(x) = \det M,$$

$$E(\det M(x))^2 = (\det M)^2 + \rho\Sigma(M^{ij})^2 + 2\rho^2\Sigma M_{ij}^2 + 6\rho^3,$$

$$E(\det M(x))^3 = (\det M)^3 + 3\rho(\det M)\Sigma(M^{ij})^2,$$
$$+ 12\rho^2(\det M)\Sigma M_{ij}^2 + 60\rho^3 \det M.$$

Thus—at least if $\det M \neq 0$—we can solve for $\Sigma(M^{ij})^2$ and ΣM_{ij}^2 as long as $\rho \neq 0$. The case of $\det M = 0$ requires the computation of the fourth moment.

In general one ends up with the determinant of M, and the sum of squares of all its $(n - k) \times (n - k)$ minors for each $k = 1, \cdots, n - 1$. It is now a simple exercise using the Cauchy-Binet formula to see that these quantities give exactly the eigenvalues of MM^*.

A complete proof of Theorem II, as well as a discussion for the case of a general correlation matrix R, will appear elsewhere.

REFERENCE

1. F. Alberto Grünbaum, *Inverse problems for nonlinear random systems*, Springer-Verlag (to appear).

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