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CLOSED ALGEBRAS OF SMOOTH FUNCTIONS

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Introduction. In this note we announce sufficient conditions for an algebra to be a subalgebra of $C^\infty(M)$ for some smooth manifold-without-boundary M. In fact, we are able to determine when M is compact and, more generally, when M carries a Riemannian structure. We maintain the notation and terminology used in [5] and [7]. In addition, $\mathfrak{m}_p$ will denote the unique maximal ideal in the stalk $\mathcal{A}_p$. We assume throughout this note that A is a geometrically homogeneous, harmonic, strongly semisimple, $\mathbf{R}$-algebra with identity. We also assume that A is strongly regular and note that, as a consequence, $\hat{f}$ is a continuous real-valued function on $\Gamma(A)$, for each $f \in A$ [1]. For the sake of brevity, we call an algebra satisfying the above conditions smooth.

Results. If $\mathfrak{m}_p$ is an $\mathcal{A}_p$-module of finite type, then we set $n \cdot \dim_A(M_p)$ equal to the minimal number of generators required for $\mathfrak{m}_p$.

DEFINITION 1. If there exists a positive integer k such that for each $M_p \in \mathfrak{M}(A)$, $n \cdot \dim_A(M_p) = k$, then we say that A has finite naive dimension k, expressed by $n \cdot \dim(A) = k$.

If $\sigma \in H^0(U, \mathcal{A})$, then by $[\sigma](p)$ we mean $[\sigma(p)] \in \mathfrak{m}_p/\mathfrak{m}_p^2$, where $p \in U$.

DEFINITION 2. A is said to be locally framed if for each $M_p \in \mathfrak{M}(A)$ there exists a local unit e_p at M_p, a relatively compact open neighborhood U of p with $p \in \bar{U} \subset u(e_p) \subset \Gamma(A)$, and sections $\sigma_1, \cdots, \sigma_k \in H^0(\Gamma(A), \mathcal{A})$ such that the family

$$([\sigma_1 - \hat{\sigma}_1(q)e_p](q), \cdots, [\sigma_k - \hat{\sigma}_k(q)e_p](q))$$

spans $\mathfrak{m}_q/\mathfrak{m}_q^2$, where $q \in \bar{U}$ and $k = n \cdot \dim(A)$. The sections $\sigma_1|_{\bar{U}}, \cdots,$

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$\sigma_k|_{\bar U} \in H^0(\bar U, A)$ will be called a system of parameters for $\bar U$.

The problem of deciding when $\Gamma(A)$ is a topological manifold is a local problem, as the following theorem demonstrates.

THEOREM 1. *If U is a relatively compact open subset of $\Gamma(A)$, then* $\max(H^0(\bar U, A)) \cong \Gamma(H^0(\bar U, A)) \cong \bar U$.

If $\sigma_1, \cdots, \sigma_k$ is a system of parameters for $\bar U$ then for each $i = 1, \cdots, k$ we may consider the ideal $I_i = (\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_k)$ in $H^0(\bar U, A)$.

DEFINITION 3. If $p \in U$ then we say that A is linearly idemfree at p if the algebra $H^0(\bar U, A)/I_i$ has no nontrivial idempotents for $i = 1, \cdots, k$. If A is linearly idemfree at each $p \in \Gamma(A)$, then we call A linearly idemfree.

THEOREM 2. *If A is a smooth, locally framed, linearly idemfree algebra, then $\Gamma(A)$ is a k-dimensional topological manifold-without-boundary.*

We denote the A-module of $\mathbf{R}$-algebra derivations of A by $\mathrm{Der}_{\mathbf{R}}(A)$.

DEFINITION 4. If A is a smooth algebra with $n \cdot \dim(A) = k$, then we say $\mathrm{Der}_{\mathbf{R}}(A)$ is tangent to A provided

(i) $\mathrm{Der}_{\mathbf{R}}(A)$ is a finitely generated projective A-module,

(ii) for each $M_p \in \mathfrak{M}(A)$, $\mathrm{Der}_{\mathbf{R}}(A)/M_p\mathrm{Der}_{\mathbf{R}}(A)$ is a real vector space of dimension k.

Some technical modifications of the methods used in [2] and [6] allow us to linearize the problem of local coordinization, provided $\mathrm{Der}_{\mathbf{R}}(A)$ is tangent to A. In fact, this linear approximation is quite adequate.

THEOREM 3. *If A is a smooth algebra with $n \cdot \dim(A) = k$ such that $\mathrm{Der}_{\mathbf{R}}(A)$ is tangent to A, then A is locally framed.*

DEFINITION 5. A is said to be a k-differentiable algebra provided

(i) A is smooth,

(ii) $n \cdot \dim(A) = k$,

(iii) $\mathrm{Der}_{\mathbf{R}}(A)$ is tangent to A,

(iv) A is linearly idemfree.

Note that if A is a k-differentiable algebra then, by Theorems 2 and 3, $\Gamma(A)$ is a topological k-manifold. Actually, using a local version of a lemma due to Banaschewski [3], we are able to prove

THEOREM 4. *If A is a k-differentiable algebra then $\Gamma(A)$ is a smooth manifold.*

DEFINITION 6. If M is a smooth manifold then a harmonic subalgebra A of $C^\infty(M)$ is said to be a closed algebra of smooth functions on M provided A contains an atlas for M and A is closed under the differential operators corresponding to this atlas.

Definition 6 enables us to sharpen Theorem 4.

THEOREM 5. *If A is a k-differentiable algebra then A is isomorphic to a closed algebra of smooth functions on $\Gamma(A)$.*

THEOREM 6. *If A is a k-differentiable algebra with $\mathfrak{G}(A) = \mathrm{Max}(A)$, then A is isomorphic to a closed algebra of smooth functions on the compact manifold $\Gamma(A)$.*

Bkouche has obtained [4] an algebraic condition which determines when an open subset of $\max(A)$ is paracompact, for certain commutative rings A. Using this characterization and the well-known fact that every paracompact smooth manifold admits a Riemannian metric we deduce

THEOREM 7. *If A is a k-differentiable algebra such that A_0 is a projective ideal, then A is isomorphic to a closed algebra of smooth functions on the Riemannian manifold $\Gamma(A)$.*

Of course, if M is a Riemannian k-manifold, then $C^\infty(M)$ is a k-differentiable algebra having $C_0^\infty(M)$, the smooth functions with compact support, as a projective ideal. A more comprehensive treatment of these results will appear elsewhere.

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