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COHEN-MACAULAY RINGS AND CONSTRUCTIBLE POLYTOPES

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We wish to point out how certain concepts in commutative algebra are of value in studying combinatorial properties of simplicial complexes. In particular, we obtain new restrictions on the f-vectors of simplicial convex polytopes.

Let Δ be a finite simplicial complex with vertex set $V = \{v_1, v_2, \cdots, v_n\}$. We call the elements of Δ the *faces* of Δ. If the largest face of Δ has d elements, then we say $\dim \Delta = d - 1$. The *f-vector* of Δ is $(f_0, f_1, \cdots, f_{d-1})$, where $\dim \Delta = d - 1$ and exactly f_i faces of Δ have $i + 1$ elements. Define for positive integers m,

$$H(\Delta, m) = \sum_{i=0}^{d-1} f_i \binom{m-1}{i}.$$

Also define $H(\Delta, 0) = 1$. We say that Δ is *constructible* [2] if it can be obtained by the following recursive procedure: (a) Every simplex is constructible, and (b) if Δ and Δ' are constructible of the same dimension d, and if $\Delta \cap \Delta'$ is constructible of dimension $d - 1$, then $\Delta \cup \Delta'$ is constructible.

We know of two main classes of constructible Δ's: (A) The boundary complex of a simplicial convex polytope is constructible. This follows from [1]. (B) Let D be a finite distributive lattice, and let D' be D with the top element and bottom element removed. Let Δ be the simplicial complex whose faces are the chains of D'. Then Δ is constructible.

If h and i are positive integers, then h can be written uniquely in the form

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$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Following McMullen [5], define

$$h^{\langle i \rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1} + 1}{i} + \cdots + \binom{n_j + 1}{j+1}.$$

Also define $0^{\langle i \rangle} = 0$.

THEOREM 1. *A vector* $(f_0, f_1, \cdots, f_{d-1})$ *of positive integers is the f-vector of some constructible* Δ *of dimension* $d-1$ *if and only if* $0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$, $1 \leq i \leq d-1$, *where* $h_1, h_2, \cdots, h_d$ *are defined by*

$$\sum_{m=0}^{\infty} H(\Delta, m)x^m = (1 + h_1 x + h_2 x^2 + \cdots + h_d x^d)/(1-x)^d.$$

In the case where Δ is the boundary complex of a simplicial convex polytope, the numbers h_i are equal to the numbers $g_{i-1}^{(d)}$ of McMullen [4]. Theorem 1 implies

$$h_i \leq \binom{f_0 - d + i - 1}{i}$$

and is therefore a strengthening of the upper bound conjecture for convex polytopes (proved in [4]), and also a generalization to constructible polytopes.

We shall indicate the main idea used to prove the "only if" part of Theorem 1. Given Δ of dimension $d-1$, let k be any field and let $R = k[v_1, v_2, \cdots, v_n]$ be the polynomial ring over k whose variables are the vertices of Δ. Define a homogeneous ideal I of R by taking for generators of I all squarefree monomials $v_{i_1} v_{i_2} \cdots v_{i_s}$ with $\{v_{i_1}, v_{i_2}, \cdots, v_{i_s}\} \notin \Delta$. Let $A_\Delta = R/I$. It is easily seen that (Krull) dim $A_\Delta = d$ and that $H(\Delta, m)$ is the Hilbert function of A_Δ. By [2, Theorem 2°], A_Δ is Cohen-Macaulay (i.e., $hd_R A_\Delta = n - d$) if Δ is constructible. The "only if" part of Theorem 1 now follows from the following elaboration and generalization of a result of Macaulay [3].

THEOREM 2. *Let* $H(m)$ *be a function from the nonnegative integers to the nonnegative integers. Let* $0 \leq r \leq d \leq n$ *be integers, and let* k *be any field. The following two conditions are equivalent.*

(i) *There is a homogeneous ideal* I *of* $R = k[x_1, x_2, \cdots, x_n]$ *such that if* $A = R/I$, *then* $\dim A = d$, $hd_R A \leqslant n - r$, *and* $H(m)$ *is the Hilbert function of* A.

(ii) $H(0) = 1$; $H(1) \leqslant n$; $H(m)$ *is a polynomial of degree* $d - 1$ *for* m *large*; *and* $0 \leqslant h_{i+1,r} \leqslant h_{i,r}^{\langle i \rangle}$, $i \geqslant 1$, *where*

$$(1 - x)^r \sum_{m=0}^{\infty} H(m)x^m = \sum_{i=0}^{\infty} h_{i,r}x^i.$$

CONJECTURE 1. If Δ is as in (A) above, then A_Δ is Gorenstein.

CONJECTURE 2. Let $H(m)$, $r = d$, n, and k be as in Theorem 2. Let $h_i = h_{i,d}$ and $l_i = h_i - h_{i-1}$, $i \geqslant 1$. The following conditions are equivalent.

(i) There is a homogeneous ideal I of $R = k[x_1, \cdots, x_n]$ such that if $A = R/I$, then $\dim A = d$, A is Gorenstein, and $H(m)$ is the Hilbert function of A.

(ii) $H(0) = 1$; $H(1) \leqslant n$; for some $t \geqslant 0$, $h_t \neq 0$ and $h_s = 0$ if $s > t$; $h_i = h_{t-i}$ for $0 \leqslant i \leqslant t$; and $0 \leqslant l_{i+1} \leqslant l_i^{\langle i \rangle}$ for $1 \leqslant i \leqslant [t/2]$.

Conjectures 1 and 2 are closely related to the main conjecture of [5].

ADDED IN PROOF. Recent work of G. Reisner implies that A_Δ is Gorenstein when $|\Delta|$ is a sphere. This establishes Conjecture 1 and also implies the previously open "upper bound conjecture for spheres."

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