BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1975

HARMONIC FORMS AND RIESZ TRANSFORMS FOR RANK ONE SYMMETRIC SPACES

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Communicated by Hans Weinberger, June 11, 1974

We study harmonic forms on a noncompact rank one symmetric space M; that is, differential forms satisfying the equations $d\omega = 0$, $\delta\omega = 0$. We define "Hardy spaces" $\mathbf{H}^p$ of harmonic forms on M and study their boundary behavior. Fractional and singular integral operators are introduced on an Iwasawa group $\overline{N}$ of M, and used to characterize the boundary values of forms in $\mathbf{H}^p$, setting up an explicit isomorphism between these spaces and the ordinary L^p spaces on $\overline{N}$. In this sense, these operators play a similar role to that of the Riesz transforms on Euclidean spaces and compact Lie groups associated to the "conjugate systems" of harmonic functions studied by Coifman, Stein, and Weiss [1], [7].

1. **Some vector fields on** M. Let G be the identity connected component of the group of isometries of M; fix an Iwasawa decomposition $G = KAN$ of G, and let $\overline{N} = \theta N$, where θ is the Cartan involution of G associated to K. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{n}, \overline{\mathfrak{n}}, \mathfrak{a}$ be the Lie algebras of the groups $G, K, N, \overline{N}, A$. Now define a right-action τ of the solvable group $\overline{S} = \overline{N}A$ on $M = G/K$ as follows: since $G = \overline{S}K$, each $x \in M$ can be written uniquely as $x = s \cdot o$, where $o = \{K\}$, $s \in \overline{S}$; then for $s' \in \overline{S}$ let $\tau(s')(s \cdot o) = ss' \cdot o$. For $X \in \overline{\mathfrak{s}}$, considered as a left invariant vector field on $\overline{S}$, define a vector field $\widetilde{X}$ on M by $\widetilde{X}_{\overline{n}a\circ o} = \tau_*(\mathrm{Ad}\,(a^{-1})X)$, $\overline{n} \in \overline{N}$, $a \in A$, where τ_* denotes the infinitesimal action of $\overline{\mathfrak{s}}$ on M induced by τ. Since the action τ is free, $X \longrightarrow \widetilde{X}$ maps a basis of $\overline{\mathfrak{s}}$ onto an everywhere defined frame of vector fields on M. Moreover, $[X, Y]^\sim = [\widetilde{X}, \widetilde{Y}]$ whenever X and Y are both in $\overline{\mathfrak{n}}$ and $[\widetilde{X}, \widetilde{Y}] = 0$ if $X \in \mathfrak{a}$. Note that the integral curves of $\widetilde{X}$, $X \in \mathfrak{a}$ are geodesics in M which are orthogonal to the family of submanifolds $\overline{N}a \cdot o$, $a \in A$.

AMS (MOS) subject classifications (1970). Primary 43A85, 53C35; Secondary 35H05.

Key words and phrases. Harmonic forms, Lie groups, fundamental solution, singular and fractional integrals.

[1]Partially supported by NSF grant GP 29704 A3.

[2]Partially supported by NSF grant GP 28448.

Let Δ_+ be the set of positive restricted roots of ad $\mathfrak{a}$ associated to the Iwasawa decomposition chosen above. Then $\bar{\mathfrak{n}} = \Sigma_{\alpha\in\Delta_+}\bar{\mathfrak{g}}_\alpha$, where $\bar{\mathfrak{g}}_\alpha$ is the root space corresponding to the root $-\alpha$, and the decomposition $\bar{\mathfrak{s}} = \mathfrak{a} + \Sigma_{\alpha\in\Delta_+}\bar{\mathfrak{g}}_\alpha$ is orthogonal with respect to the inner product $(X, Y) = -B(X, \theta Y)$, with B denoting the Killing form of $\mathfrak{g}$. Set $\rho = \frac{1}{2}\Sigma_{\alpha\in\Delta_+}m_\alpha\alpha$, where m_α is the multiplicity of the root α.

Note, in general, $\delta\omega = 0 = d\omega$ does not imply that the functions $\omega(\widetilde{X})$, $X \in \bar{\mathfrak{s}}$, are annihilated by the Laplace-Beltrami operator Δ. The following result, however, provides an opening for the application of Fatou type theorems to the present context:

PROPOSITION 1. *If* $d\omega = 0 = \delta\omega$ *and* X *belongs to the center of* $\bar{\mathfrak{n}}$, *then* $\Delta\omega(\widetilde{X}) = 0$.

2. **Riesz transforms on $\bar{N}$.** Henceforth we assume that rank $M = 1$. We can then choose a root $\alpha \in \Delta_+$ such that either $\Delta_+ = \{\alpha\}$ or $\Delta_+ = \{\alpha, 2\alpha\}$, so that $\bar{\mathfrak{n}} = \bar{\mathfrak{g}}_\alpha + \bar{\mathfrak{g}}_{2\alpha}$. Set $m = m_\alpha + 2m_{2\alpha}$; in order to avoid technical complications it will be convenient to assume that $m \geqslant 3$ (see the remark at the end).

If $X \in \mathfrak{g}$, let $|X| = (X, X)^{1/2}$. Now, for $\epsilon \geqslant 0$, define the function

$$\langle\bar{n}\rangle_\epsilon = (\eta|X|^2 + \epsilon^2)^2 + |Y|^2,$$

where $\bar{n} \in \bar{N}$, $\bar{n} = \exp(X + Y)$, $X \in \bar{\mathfrak{g}}_\alpha$, $Y \in \bar{\mathfrak{g}}_{2\alpha}$; η denotes the constant $(\alpha, \alpha)/8 = (16(m_\alpha + 4m_{2\alpha}))^{-1}$.

Consider now the differential operator $D = \Sigma_i X_i^2$ on $\bar{N}$, where $X_1, X_2, \cdots, X_{m_\alpha}$ is an orthonormal basis of $\bar{\mathfrak{g}}_\alpha$; this operator is clearly independent of the choice of the basis $\{X_i\}$.

PROPOSITION 2. $D\langle\bar{n}\rangle_\epsilon^{-(m-2)/4} = -(m-2)m_\alpha\eta\epsilon^2\langle\bar{n}\rangle_\epsilon^{-(m+2)/4}$.

The proof is a straightforward computation using the following

LEMMA (arbitrary rank). *Let* β *be a restricted root and* $\{X_i\}$ *an orthonormal basis of* $\mathfrak{g}_\beta$. *Then, for* $X \in \mathfrak{g}_\beta$, $Y \in \mathfrak{g}_{2\beta}$

(i) $\Sigma_i|[X, X_i]|^2 = 2m_{2\beta}(\beta, \beta)|X|^2$;

(ii) $\Sigma_i(Y, [X, X_i])^2 = 2(\beta, \beta)|X|^2|Y|^2$.

Now put

$$c^{-1} = -m_\alpha(m-2)\eta \cdot \int_{\bar{N}} \langle\bar{n}\rangle_1^{-(m+2)/4}\, d\bar{n}$$

and define the function

$$G(\bar{n}) = c\langle \bar{n}\rangle_0^{-(m-2)/4}$$

on $\bar{N}$.

PROPOSITION 3. *$G(\bar{n})$ is a fundamental solution for D.*

REMARKS. (1) The operator D is not elliptic except, of course, when $\bar{\mathfrak{g}}_{2\alpha} = (0)$. Proposition 3 shows, however, that D is always hypoelliptic. This is also a consequence of the fact that the root space $\bar{\mathfrak{g}}_\alpha$ generates the Lie algebra $\bar{\mathfrak{n}}$ (see [3]).

(2) For the Heisenberg group this fundamental solution was obtained by Folland [2].

Now, for each element $Z \in \bar{\mathfrak{n}}$, define the associated "Riesz kernel" r_Z and the "Riesz transform" R_Z on $\bar{N}$ by

$$r_Z(\bar{n}) = ZG(\bar{n}), \qquad R_Z f(\bar{n}) = f * r_Z(\bar{n}) = \int_{\bar{N}} f(g) r_Z(g^{-1}\bar{n})\, dg.$$

A straightforward computation gives, for $Z \in \bar{\mathfrak{g}}_\alpha$,

$$r_Z(\bar{n}) = \tfrac{1}{2}(2-m)c\langle \bar{n}\rangle_0^{-(m+2)/4} \cdot (2\eta|X|^2(X, Z) + 1/2(Y, [X, Z])),$$

and for $Z \in \bar{\mathfrak{g}}_{2\alpha}$,

$$r_Z(\bar{n}) = \tfrac{1}{2}(2-m)c\langle \bar{n}\rangle_0^{-(m+2)/4}(Y, Z),$$

where $\bar{n} = \exp(X + Y)$, $X \in \bar{\mathfrak{g}}_\alpha$, $Y \in \bar{\mathfrak{g}}_{2\alpha}$.

THEOREM 1. *The Riesz transform R_Z is*

(i) *a bounded operator from $L^p(\bar{N})$ into $L^p(\bar{N})$, $1 < p < \infty$, if $Z \in \bar{\mathfrak{g}}_{2\alpha}$, and*

(ii) *a bounded operator from $L^p(\bar{N})$ into $L^q(\bar{N})$, $1 < p < q < \infty$, $q^{-1} = p^{-1} - m^{-1}$, if $Z \in \bar{\mathfrak{g}}_\alpha$.*

For the proof, observe that the functions r_Z are singular integral kernels (if $Z \in \bar{\mathfrak{g}}_{2\alpha}$) and fractional integral kernels (if $Z \in \bar{\mathfrak{g}}_\alpha$). They satisfy the following homogeneity properties under the "dilations" of $\bar{N}$ induced by A:

$$r_Z(a^{-1}na) = e^{-(m-1)\alpha(\log a)} \cdot r_Z(\bar{n}) \qquad (Z \in \bar{\mathfrak{g}}_\alpha),$$

$$r_Z(a^{-1}na) = e^{-m\alpha(\log a)} \cdot r_Z(\bar{n}) \qquad (Z \in \bar{\mathfrak{g}}_{2\alpha}),$$

and are odd for Z in $\bar{\mathfrak{g}}_{2\alpha}$.

General theorems about these type of operators on nilpotent groups have also been obtained by Korányi-Vági [5] and Stein [6].

Observe that the Riesz transforms satisfy the following relations:

$$XR_Y - YR_X = R_{[X, Y]} \qquad (X, Y \in \overline{\mathfrak{n}}),$$

$$\sum_i X_i \circ R_{X_i} = \text{identity} \quad (\{X_i\} \text{ an orthonormal basis of } \overline{\mathfrak{g}}_\alpha).$$

3. **Boundary values of harmonic forms.** Let H be the unique element in $\mathfrak{a}$ such that $\alpha(H) = 1$, and define the vector field $\widetilde{W}$ on M by

$$\widetilde{W}_{\overline{n}a \cdot o} = e^{2\alpha(\log a)} \widetilde{H}_{na \cdot o} \qquad (n \in \overline{N}, a \in A).$$

For each p, $1 < p < m$, let $\mathbf{H}^p$ be the space of all 1-forms ω on M such that:

(a) $d\omega = 0 = \delta\omega$,

(b) $\omega(\widetilde{W}), \omega(\widetilde{Y}) \in L^p \quad (Y \in \overline{\mathfrak{g}}_{2\alpha})$,

(c) $\omega(\widetilde{X}) \in L^q \quad (X \in \overline{\mathfrak{g}}_\alpha,\ q^{-1} = p^{-1} - m^{-1})$.

With $\overline{N}$ regarded as a boundary for the symmetric space M (see e.g. [4]), we can now state the following results.

THEOREM 2. *Let $f \in L^p(\overline{N})$, $1 < p < m$, and define a 1-form ω_f on M by letting $\omega_f(\widetilde{W}) = f * \widetilde{W}(P * G)$, $\omega_f(\widetilde{X}) = f * P * r_X$, $X \in \overline{\mathfrak{n}}$. ($P$ is the Poisson kernel, G is the fundamental solution of D, r_X the kernel constructed above.) Then*

(i) $\omega_f \in \mathbf{H}^p$.

(ii) *$\omega_f(\widetilde{W})$ converges in L^p and almost everywhere to f.*

(iii) *$\omega_f(\widetilde{X})$ converges in L^q (respectively L^p) and almost everywhere to $f * r_X$ if $X \in \overline{\mathfrak{g}}_\alpha$ (respectively $X \in \overline{\mathfrak{g}}_{2\alpha}$).*

THEOREM 3. *If $\omega \in \mathbf{H}^p$, then there exists a (unique) $f \in L^p(\overline{N})$ such that $\omega = \omega_f$.*

REMARK. In order to obtain analogous results when $m < 3$ (real hyperbolic spaces of dimension 2 or 3), one must modify the definition of the corresponding $\mathbf{H}^p$ spaces and Riesz transforms; in dimension 2 this leads to the classical connection between conjugate harmonic functions in the upper half-plane and the Hilbert transform in the line.

A detailed version of these results will appear elsewhere.

REFERENCES

1. R. R. Coifman and G. Weiss, *Invariant systems of conjugate harmonic functions associated with compact Lie groups*, Studia Math. **44** (1972).

2. G. B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79** (1973), 373–376. MR **47** #3816.

3. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171. MR **36** #5526.

4. A. Korányi, *Boundary behavior of Poisson integrals on symmetric spaces*, Trans. Amer. Math. Soc. **140** (1969), 393–409. MR **39** #7132.

5. A. Korányi and S. Vági, *Singular integrals on homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa **25** (1971).

6. E. M. Stein, *Singular integrals and estimates for the Cauchy-Riemann equations*, Bull. Amer. Math. Soc. **79** (1973), 440–445. MR **47** #3851.

7. E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables*. I. *The theory of HP-spaces*, Acta Math. **103** (1960), 25–62. MR **22** #12315.

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