BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 1, January 1975

L_2-REPRESENTATIONS AND A PLANCHEREL-TYPE THEOREM FOR PARABOLIC SUBGROUPS

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Communicated by S. S. Chern, July 18, 1974

Let G be a semisimple Lie group with Iwasawa decomposition $G = KAN$. In this note we give a precise condition for the existence of square-integrable representations of the nilpotent subgroup N. In that case we write down a Plancherel formula for the solvable subgroup AN. Full details and complete proofs will appear in a later paper.

These results are in essence part of the author's doctoral dissertation [1]. He would like to thank Professor Joseph A. Wolf for his patient advice and encouragement.

I. **L_2-representations of the nilpotent subgroup** N. Let N be a unimodular locally compact group. Let π be an irreducible unitary representation of N on a Hilbert space $H(\pi)$. Then π is square-integrable (or L_2) if there are nonzero vectors x_1 and x_2 in $H(\pi)$ such that

$$\int_{N/Z} |(\pi(s)x_1, x_2)|^2 \, d\mu(s) < \infty$$

where Z is the center of N and $d\mu(s)$ denotes integration over N/Z with respect to a Haar measure μ on N/Z.

If N is a connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, let Z and $\mathfrak{z}$ be the respective centers on N and $\mathfrak{n}$. Let $\mathfrak{n}^*$, $\mathfrak{z}^*$ be the respective linear duals of $\mathfrak{n}$, $\mathfrak{z}$. Define an alternating bilinear form b_f on $\mathfrak{n}$ by $b_f(x, y) = f([x, y])$ for $f \in \mathfrak{n}^*$ and $[\ ,\ ]$ the multiplication for $\mathfrak{n}$. If $f \in \mathfrak{z}$, we can extend f trivially to $\mathfrak{n}$ and define b_f on $\mathfrak{n}/\mathfrak{z}$. Moore and Wolf [3] have shown the following:

PROPOSITION 1. *N has L_2-representations if and only if there exists an $f \in \mathfrak{z}^*$ such that b_f is nondegenerate on $\mathfrak{n}/\mathfrak{z}$.*

AMS (MOS) subject classifications (1970). Primary 22D10, 22E25, 22E45, 43A30, 43A65.

Key words and phrases. Parabolic subgroup, semisimple Lie group, Plancherel theorem, nonunimodular group, square-integrable representations, solvable subgroups of type AN.

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Then a straightforward argument (see [1]) gives the following condition.

THEOREM 1. *If there is a noncentral* x *in* $\mathfrak{n}$ *such that* $[x, \mathfrak{n}] \cap \mathfrak{z} = \{0\}$, *then* N *has no* L_2*-representations.*

We note that b_f is a skew-symmetric bilinear form on $\mathfrak{n}/\mathfrak{z}$ and hence we can define the Pfaffian $\mathrm{Pf}(f)$ to be the Pfaffian of b_f. Proposition 1 can be restated.

PROPOSITION 1$'$. N *has* L_2*-representations if and only if there exists an* $f \in \mathfrak{z}^*$ *such that* $\mathrm{Pf}(f) \neq 0$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. Let $\mathfrak{M}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. For $\alpha: \mathfrak{a} \longrightarrow \mathbf{R}$, $\alpha \neq 0$ a real linear functional on $\mathfrak{a}$, let

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [a, x] = \alpha(a)x, \text{ for all } a \in \mathfrak{a}\}.$$

If $\mathfrak{g}^\alpha \neq \{0\}$, $\mathfrak{g}^\alpha$ is the $\mathfrak{a}$-root space of $\mathfrak{g}$ for α, and α is an $\mathfrak{a}$-root of $\mathfrak{g}$. Define $\mathfrak{g}^0 = \mathfrak{M} + \mathfrak{a}$.

THEOREM 2. *Let* α *be an* $\mathfrak{a}$*-root. Then* $\mathfrak{g}^0$ *acts irreducibily over* $\mathbf{R}$ *on* $\mathfrak{g}^\alpha$ *by the adjoint action.*

THEOREM 3. *Let* β, γ, *and* $\beta + \gamma$ *be nonzero* $\mathfrak{a}$*-roots with* β *and* γ *linearly independent. Then* $[\mathfrak{g}^\beta, \mathfrak{g}^\gamma] = \mathfrak{g}^{\beta+\gamma}$.

There is a positive $\mathfrak{a}$-root system $\Sigma_{\mathfrak{a}}^+$ such that $\mathfrak{n} = \Sigma\{\mathfrak{g}^\gamma \mid \gamma \in \Sigma_{\mathfrak{a}}^+\}$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$.

THEOREM 4. *Let* μ *be the maximal* $\mathfrak{a}$*-root. Then* $\mathfrak{g}^\mu = \mathfrak{z}$.

This enables us to find a condition for the existence of L_2-representations of N.

THEOREM 5. *Let* G *be a simple Lie group with Iwasawa decomposition* $G = \mathrm{KAN}$ *and corresponding Lie algebra decomposition* $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. *Then* N *has square-integrable representations if and only if the extended Dynkin diagram of the reduced* $\mathfrak{a}$*-root system is of type* A_1 *or of type* A_2.

Theorems 1 through 4 can be used to show that the condition on the root system is necessary for the existence of L_2-representations. Sufficiency can be shown by explicitly calculating the Pfaffian in these cases. Details of these calculations will appear later.

II. The Plancherel formula in the rank 1 case. Here G is a simple Lie group with Iwasawa decomposition $G = KAN$ and G/K has symmetric space rank 1. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the corresponding Lie algebra decomposition. Let $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q}$, or Cay be one of the real division algebras; here $\mathbf{Q}$ is the quaternions and Cay the real Cayley division algebra. Let $\mathrm{Im}(\mathbf{F})$ be the imaginary part of $\mathbf{F}$, that is, the orthogonal complement to $\mathbf{R}$ in the usual orthonormal basis. Then $\mathfrak{g}$ can be realized as $so(1, k; \mathbf{F})$, for $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or $\mathbf{Q}$, or as the real form of the exceptional Lie algebra of type F_4 with maximal compact subalgebra $so(9)$, that is, with Cartan index -20. Then $\mathfrak{n}$ can be realized as $\mathfrak{n}_n(\mathbf{F}) = \mathbf{F}^n + \mathrm{Im}(\mathbf{F})$ where $\mathfrak{n}_n(\mathbf{F})$ is the nilpotent part of $so(1, n+1; \mathbf{F})$, or, if $n = 1$ and $\mathbf{F} = \mathit{Cay}$, of $F_{4(-20)}$. The multiplication for $\mathfrak{n}_n(\mathbf{F})$ is given by making $\mathrm{Im}(\mathbf{F})$ the center of $\mathfrak{n}_n(\mathbf{F})$ and, for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbf{F}^n$, defining

$$[x, y] = \mathrm{Im}\left(\sum_{i=1}^{n} x_i \bar{y}_i\right) \quad \text{in } \mathrm{Im}(\mathbf{F}).$$

Then, in particular, $[\mathfrak{n}_n(\mathbf{F}), \mathfrak{n}_n(\mathbf{F})] \subset \mathrm{Im}(\mathbf{F}) = \mathfrak{z}(\mathfrak{n})$.

Using an idea due to C. C. Moore [2], we are able to find a "Plancherel Theorem" for AN (or, more accurately, for NA) in this case.

THEOREM 6. *Let* $\mathfrak{n} = \mathfrak{n}_n(\mathbf{F})$ *for* $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{Q}$, *or Cay be given as above. Let* $\mathfrak{a} = \mathbf{R}$ *be the (vector) abelian Lie algebra. Let* N, A *be the connected Lie groups of* $\mathfrak{n}, \mathfrak{a}$ *respectively, and* NA *their semidirect product and hence the Lie algebra of* $\mathfrak{n} + \mathfrak{a}$. *Let* $\dim \mathfrak{z}(\mathfrak{n}) = k$. *Then, for* $\gamma \in C_c^\infty(NA)$,

$$\gamma(1_{NA}) = \int_{S^{k-1}} \mathrm{trace}\ \pi_\lambda(D\gamma)\, d\sigma(\lambda)$$

where $d\sigma(\lambda)$ *is Lebesgue measure on the unit sphere in* $\mathfrak{z}(\mathfrak{n})^*$, π_λ *is in the unitary dual of* NA *parametrized by* λ, *and* D *is an operator on* $C_c^\infty(\mathbf{R}^k)$ *given by*

$$D = (i/2\pi)(\Delta)^{1/2} \quad \text{for } \mathbf{F} = \mathbf{R},$$

$$D = (i/2\pi)^q (\Delta)^{q/2} \quad \text{for } \mathbf{F} = \mathbf{C}, \mathbf{Q}, \text{ or } \mathit{Cay},$$

where $q = (n(k+1) + 2k)/2$ *and* Δ *is the Laplacian operator on* $C_c^\infty(\mathbf{R}^k)$ *and* $\mathbf{R}^k$ *is the center of* N.

BIBLIOGRAPHY

1. F. W. Keene, *Square-integrable representations of Iwasawa subgroups of a semisimple Lie group*, Doctoral Dissertation, University of California, Berkeley, Calif., 1974.

2. C. C. Moore, *Representations of solvable and nilpotent groups and harmonic analysis on nil and solvmanifolds*, Proc. Sympos. Pure Math., vol. 26, Amer. Math. Soc., Providence, R. I., 1973, pp. 3–44.

3. C. C. Moore and J. A. Wolf, *Square integrable representations of nilpotent groups*, Trans. Amer. Math. Soc. 185 (1973), 445–462.

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