BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

A SEPARABLE SOMEWHAT REFLEXIVE BANACH SPACE WITH NONSEPARABLE DUAL

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Communicated by Jacob Feldman, November 18, 1973

ABSTRACT. An example is given of a separable Banach space X whose dual is not separable, but each infinite-dimensional subspace of X contains an infinite-dimensional subspace isomorphic to Hilbert space. Thus X contains no subspace isomorphic to c_0 or l_1, X is somewhat reflexive, and no nonreflexive subspace has an unconditional basis.

It has been conjectured that every infinite-dimensional Banach space has an infinite-dimensional subspace that is either reflexive or isomorphic to c_0 or to l_1 [**9**, p. 165]. A counterexample would also be an example of a space that has no infinite-dimensional subspace with an unconditional basis [**6**, Theorem 2, p. 521]. It is known that there is a nonreflexive Banach space J with no subspace isomorphic to c_0 or to l_1 [**6**, pp. 523–527], but J^{**} is separable. Each of the following is a necessary and sufficient condition for a separable Banach space X to contain a subspace isomorphic to l_1; separability is not needed for conditions (i) and (ii) (see [**5**, Theorem 2.1, p. 13] and [**10**, p. 475]).

(i) $L_1[0, 1]$ is isomorphic to a subspace of X^*.

(ii) $C[0, 1]^*$ is isomorphic to a subspace of X^*.

(iii) X^* has a subspace isomorphic to $l_1(\Gamma)$ for some uncountable Γ.

A natural and well-known conjecture in view of the preceding is that a Banach space has a subspace isomorphic to l_1 if the space is separable and its dual is not separable (e.g., see [**1**, §9, p. 243], [**2**, §5.4, p. 174], and the last paragraph of [**11**]). It will be shown that this conjecture is false. The counterexample $\tilde{X}$ has the property that each infinite-dimensional subspace has an infinite-dimensional subspace isomorphic to Hilbert space. Thus $\tilde{X}$ is also a counterexample to the conjecture that each separable somewhat-reflexive space has a separable dual (see [**3**, Problem 3, p. 191] and [**8**, Remark IV.2, p. 86]). Also, neither c_0 nor l_1 has an infinite-dimensional subspace isomorphic to Hilbert space, so no nonreflexive subspace has an unconditional basis [**6**, Theorem 2, p. 521]. It has been

AMS (MOS) subject classifications (1970). Primary 46B10.

[1] Supported in part by NSF Grant GP-28578.

shown by J. Lindenstrauss and C. Stegall that X is a counterexample for several other conjectures. They will present these results in a later paper, as well as giving another example of a separable space with nonseparable dual that has no subspace isomorphic to l_1 (this space has a subspace isomorphic to c_0).

The counterexample is intimately related to the space J mentioned above, e.g., many complemented subpsaces are isometric to J. This follows from the fact that, if $x=\{x_i\} \in J$, then

$$\|x\| = \sup\left\{\left[\sum_{i=1}^{n-1}(x_{p(i+1)} - x_{p(i)})^2\right]^{1/2} : 1 \leqq p(1) < \cdots < p(n),\ n \geqq 1\right\},$$

but if x is written as $\{\xi_i\}$, where $\xi_i=x_i-x_{i+1}$ and $x=\sum_1^\infty \xi_i e_i$ with e_n the sequence for which $x_i=1$ if $i\leqq n$ and $x_i=0$ if $i>n$, then

$$\|x\| = \sup\left\{\left[\sum_{j=1}^{n-1}\left(\sum_{i=p(j)}^{p(j+1)-1} \xi_i\right)^2\right]^{1/2} : 1 \leqq p(1) < \cdots < p(n),\ n \geqq 1\right\}.$$

To describe the counterexample, first choose a set Ω of cardinality c, each of whose members can be thought of as an infinite subset of the positive integers or as a nested sequence of intervals obtained as follows. Associate $1=b(1, 1)$ with the interval $[0, 1]$, then $2=b(2, 1)$ with $[0, \frac{1}{3}]$ and $3=b(2, 2)$ with $[\frac{2}{3}, 1]$, and in general for each positive integer n use the integers from 2^{n-1} to 2^n-1, or $\{b(n, i): 1\leqq i\leqq 2^{n-1}\}$, to label the 2^{n-1} intervals remaining at the nth stage of the process customarily used to describe the Cantor set. Now let each number t in the Cantor set determine a member of Ω, namely, the set of all integers associated with intervals containing t. By a *segment* we shall mean a finite increasing sequence (possibly empty) of consecutive members of some set $\mathscr{A} \in \Omega$. If $\mathscr{A}\neq\mathscr{B}$ and $\mathscr{A}$ and $\mathscr{B}$ are in Ω, then $\mathscr{A}\cap\mathscr{B}$ is a nonempty initial segment of both $\mathscr{A}$ and $\mathscr{B}$. A *branch point of order k* for Ω is one of the integers $\{b(k, i): 1\leqq i\leqq 2^{k-1}\}$ that is the kth term of some member of Ω. A *branch of order k* (or a *k-branch*) is an infinite increasing sequence of consecutive members of some set $\mathscr{A} \in \Omega$ whose first member is a branch point of order k. For each sequence $x=\{x_i\}$ of real numbers with finite support, let

$$\|x\| = \sup\left\{\left[\sum_n\left(\sum_{i\in A(n)} x\right)^2\right]^{1/2}\right\},$$

where the sup is over all finite sets $\{A(n): 1\leqq n\leqq p\}$ of pairwise disjoint segments. Let $\tilde{X}$ be the completion with respect to this norm of the normed linear space X of all such sequences. Then $\tilde{X}$ is a separable Banach space.

For each $\mathscr{A} \in \Omega$, define a linear functional $f_{\mathscr{A}}$ on X by letting $f_{\mathscr{A}}(x)=\sum_{i\in\mathscr{A}} x_i$ if $x \in X$, and extending to $\tilde{X}$ by use of continuity. Then $\|f_{\mathscr{A}}\|=1$.

Also, if $m \in \mathscr{A}-\mathscr{B}$, $n \in \mathscr{B}-\mathscr{A}$, and x is the sequence $\{x_i\}$ with $x_m=1$, $x_n=-1$ and $x_i=0$ otherwise, then $(f_{\mathscr{A}}-f_{\mathscr{B}})(x)=2$ and $\|x\|=2^{1/2}$. Thus $\|f_{\mathscr{A}}-f_{\mathscr{B}}\| \geqq 2^{1/2}$ and $\tilde{X}^*$ is not separable.

THEOREM. *If $\theta>\sqrt{2}$, then each infinite-dimensional subspace of $\tilde{X}$ contains an infinite-dimensional subspace H for which there is an inner-product norm $|||\ |||$ such that*

$$|||x||| \leqq \|x\| \leqq \theta\, |||x||| \quad \text{if } x \in H.$$

PROOF. It is sufficient to prove the theorem for X. Let Y be an infinite-dimensional subspace of X and let Y^k be the subspace of Y whose members are zero at each of the finite set of branch points with order less than k. Then Y^k has finite codimension as a subspace of Y. For each x in X, let

$$[x]_k = \sup\left\{\left[\sum_n \left(\sum_{i\in B(n)} x_i\right)^2\right]^{1/2}\right\},$$

where the sup is over all sets $\{B(n)\}$ of pairwise disjoint k-branches. Let

$$\omega = \lim_{k\to\infty} \inf\{[x]_k : x \in Y^k \text{ and } \|x\| = 1\}.$$

It will be shown that $\omega=0$. Suppose $\omega>0$. For $\varepsilon>0$, choose K so that

$$\text{(1)} \qquad \inf\{[x]_k^2 : x \in Y^k \text{ and } \|x\| = 1\} > \omega^2 - \varepsilon \quad \text{if } k \geqq K.$$

Choose an increasing sequence of integers $\{m(k)\}$ with $m(1)=K$, and then a sequence $\{y^k\}$ in X such that, for each k, $\|y^k\|=1$, y^k has nonzero terms only at branch points with orders in the interval $[m(k), m(k+1))$, and

$$\text{(2)} \qquad [y^k]_{m(k)}^2 < \omega^2 + \varepsilon.$$

It will be shown that a contradiction is obtained if ε is sufficiently small. Let $y^k=\{y_i^k\}$. Since $[y^k]_K^2>\omega^2-\varepsilon$ and $y^k \in Y^{m(k)}$, there are 2^{K-1} branch points of order $m(k)$, which will be denoted by $\{b(k, p_i^k): 1\leqq i\leqq 2^{K-1}\}$ rather than using $b(m(k), p_i^k)$, and 2^{K-1} branches $\{B(k, p_i^k): 1\leqq i\leqq 2^{K-1}\}$ of order $m(k)$ starting at these branch points, such that

$$\text{(3)} \qquad \sum_i \left(\sum_{j\in B(k,p_i^k)} y_j^k\right)^2 > \omega^2 - \varepsilon.$$

Now for each i, k and κ with $i\leqq 2^{K-1}$ and $k<\kappa$, let $\sigma(k, \kappa; i)$ be j if there exists $j\leqq 2^{K-1}$ such that $b(k, p_i^k)$ and $b(\kappa, p_j^\kappa)$ are on the same K-branch. Then $\sigma(k, \kappa; i)$ is strictly increasing as a function of i and determines a one-to-one mapping of a subset of $\{p_i^k : i\leqq 2^{K-1}\}$ onto a subset of $\{p_i^\kappa : i\leqq 2^{K-1}\}$. Choose a sequence of positive integers I_1 so that if k and κ are in I_1 and $k<\kappa$, then $\sigma(k, \kappa; i)=\sigma(k; i)$ is independent of κ for each i.

Then choose a subsequence I_2 of I_1 so that if $k \in I_2$ then $\sigma(k; i)=\sigma(i)$ is independent of k for each i. Now, $\sigma[\sigma(i)]=\sigma(i)=i$ and, for each $i \leqq 2^{K-1}$, either i is in the domain of σ and there is a K-branch that contains all $b(k, p_i^k)$ for $k \in I_2$, or i is not in the domain of σ and no K-branch that contains $b(k, p_i^k)$ for some $k \in I_2$ can contain any $b(\kappa, p_j^\kappa)$ for $\kappa \neq k$ and $\kappa \in I_2$.

Now choose a subsequence I_3 of I_2 such that, for each i in the domain of σ and any two members k and κ of I_3,

$$\left|\sum_{i \in B} y_i^k - \sum_{i \in B} y_i^\kappa\right| < 2^{-K/2}\varepsilon, \tag{4}$$

where B is the K-branch containing all $b(k, p_i^k)$ for $k \in I_3$. For a λ to be chosen later, let $\{\mu(j): 1 \leqq j \leqq \lambda\}$ be any λ consecutive members of I_3 and, for any K-branch B, consider

$$\left[\sum_{i \in B}\left(\sum_{j=1}^{\lambda}(-1)^j y_i^{\mu(j)}\right)\right]^2 = \left[\sum_{j=1}^{\lambda}(-1)^j\left(\sum_{i \in B} y_i^{\mu(j)}\right)\right]^2. \tag{5}$$

For each $\mu(j)$, let $\sum_{i \in B} y_i^{\mu(j)}$ be denoted by $\rho_B^{\mu(j)}$ or $\Delta_B^{\mu(j)}$ accordingly as B contains one of the branch points $\{b[\mu(j), p_i^{\mu(j)}]\}$ or B does not contain any such branch point. Then either there exists $\iota \leqq 2^{K-1}$ and $\kappa > 0$ such that B contains $\{b(\mu(j), p_\iota^{\mu(j)}): j \leqq \kappa\}$ and B contains no other $b(\mu(j), p_i^{\mu(j)})$ for $\kappa < j \leqq \lambda$ and $i \leqq 2^{K-1}$, or else B contains at most one of $\{b[\mu(j), p_i^{\mu(j)}]: 1 \leqq j \leqq \lambda, i \leqq 2^{K-1}\}$. For any real numbers $\{a_i\}$,

$$\left(\sum_1^n a_i\right)^2 \leqq \sum_1^n 2^i a_i^2. \tag{6}$$

Therefore it follows from (4) that the expression (5) is not greater than

$$2(\rho_B^\mu)^2 + 4\left(\left[\frac{\kappa}{2}\right]2^{-K/2}\varepsilon\right)^2 + \sum_{j=1}^{\lambda} \varepsilon_j 2^\lambda (\Delta_B^{\mu(j)})^2, \tag{7}$$

where $\mu \in \{\mu(j): j \leqq \lambda\}$ (except that the first term in (7) may be missing), κ is the largest integer such that $\kappa \leqq \lambda$ and B contains $b(\mu(j), p_\iota^{\mu(j)})$ for some ι and for all $j \leqq \kappa$, and each ε_j is 0 or 1. Note that if we sum terms of type $(\Delta_{B(n)}^{\mu(j)})^2$ over any 2^{K-1} pairwise disjoint K-branches $B(n)$, then it follows from (2) and (3) that this sum is not greater than 2ε; also, there are then at most 2^{K-1} terms of the type of the first term in (7), so these can contribute to $[y^{\mu(j)}]^2_{m[\mu(j)]}$ for at most 2^{K-1} values of j. Therefore the sum of (5) or (7) over any 2^{K-1} pairwise disjoint K-branches is not greater than

$$2 \cdot 2^{K-1}(\omega^2 + \varepsilon) + 4[\lambda/2]\varepsilon^2 + \lambda \cdot 2^\lambda(2\varepsilon) \leqq 2^K\omega^2 + \varepsilon(2^K + \lambda \cdot 2^{\lambda+1}) + 2\lambda\varepsilon^2.$$

Since $\|\sum_{j=1}^{\lambda}(-1)^j y^{\mu(j)}\|^2 \geqq \lambda$, this contradicts (1) if

$$2^K\omega^2 + \varepsilon(2^K + \lambda \cdot 2^{\lambda+1}) + 2\lambda\varepsilon^2 < \lambda(\omega^2 - \varepsilon).$$

This inequality can be satisfied by choosing $\lambda > 2^K$ and then choosing ε small enough.

This concludes the proof that $\omega = 0$. Since $\omega = 0$, we can let ε be a positive number and choose an increasing sequence of integers $\{n(k)\}$ and a sequence $\{y^k\}$ in X, such that, for each k, $\|y^k\| = 1$, y^k has nonzero terms only at branch points with orders in the interval $(n(k), n(k+1))$, and

$$[y^k]^2_{n(k)} < 2^{-k}\varepsilon^2. \tag{8}$$

Let $\{a_i\}$ be a finite sequence of real numbers with $\sum a_i^2 > 0$. Then $\|\sum a_j y^j\|^2 \geqq \sum a_i^2$. Choose a finite set $\{A(n): 1 \leqq n \leqq p\}$ of pairwise disjoint segments such that

$$\left\|\sum a_j y^j\right\|^2 = \sum_n \left(\sum_{i \in A(n)} \sum_j a_j y^j_i\right)^2.$$

If A is any of these segments, then A is the union of an initial and a terminal segment, each of which contains a part of the piece of a branch between branch points of order $n(j)$ and branch points of order $n(j+1)$ for some j, and several interior segments, each of which has the property that there is a j such that the segment contains all of the piece of a branch between branch points of order $n(j)$ and branch points of order $n(j+1)$. A sum $\sum_j a_j y^j_i$ over those i in an initial segment or a sum over those i in a terminal segment contributes only to the norm of the corresponding $a_j y^j$, while a sum over an interior segment contributes to $[a_j y^j]_{n(j)}$ only. Now we can use the fact that

$$(a + b + c)^2 \leqq (2 + \varepsilon)(a^2 + b^2) + (1 + 2/\varepsilon)c^2$$

for any real numbers a, b and c, and then (6) and (8), to obtain

$$\begin{aligned}\left\|\sum a_j y^j\right\|^2 &\leqq (2 + \varepsilon)\sum a_j^2 + \left(1 + \frac{2}{\varepsilon}\right)\sum_j 2^j a_j^2 [y^j]^2_{n(j)} \\ &< (2 + \varepsilon)\sum a_j^2 + (2\varepsilon + \varepsilon^2)\sum a_j^2 = (2 + 3\varepsilon + \varepsilon^2)\sum a_j^2.\end{aligned}$$

Since ε was arbitrary, for any $\theta > \sqrt{2}$ there is an infinite sequence $\{y^k\}$ of members of X such that, for all sequences $\{a_i\}$ of real numbers,

$$\left(\sum_1^\infty a_i^2\right)^{1/2} \leqq \left\|\sum_1^\infty a_i y^i\right\| \leqq \theta\left(\sum_1^\infty a_i^2\right)^{1/2}.$$

ERRATUM ADDED IN PROOF. The sequences $\{m(k)\}$ and $\{y^k\}$ should be chosen simultaneously so that each y^k is in Y and the branches $\{B(k, p_i^k)$: $1 \leqslant i \leqslant 2^{K-1}\}$ are pieces of pairwise disjoint K-branches; $\{\mu(j)\colon 1 \leqslant j \leqslant \lambda\}$ should not be consecutive members of I_3, but chosen so that, for each branch point b of order K,

$$\sup\left\{\left[\sum_{i\in B} y_i^{\mu(j)}\right]^2\right\} < \sup\left\{\left[\sum_{i\in B} y_i^{\mu(1)}\right]^2\right\} + 2^{1-K}\epsilon,$$

where $1 \leqslant j \leqslant \lambda$ and B is any K-branch containing b. In the first term of (7), ρ_B^μ should be replaced by the sum of the absolute values of two such terms; in the next two inequalities, $2^K(\omega^2 + \epsilon)$ can now be replaced by $8\omega^2 + 16\epsilon$ and λ need not depend on K.

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