BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 2, March 1974

PARAMETRICES AND ESTIMATES FOR THE $\bar{\partial}_b$ COMPLEX ON STRONGLY PSEUDOCONVEX BOUNDARIES

BY G. B. FOLLAND AND E. M. STEIN

Communicated May 21, 1973

0. **Introduction.** Here we briefly sketch the background of the problem to be considered, and refer to Folland-Kohn [4] for definitions and proofs.

Let X be the boundary of a strongly pseudoconvex region in a complex manifold of complex dimension $n+1$, or more generally a real manifold of dimension $2n+1$ with a strongly pseudoconvex partially complex structure. We then have the tangential Cauchy-Riemann complex

$$0 \longrightarrow \Lambda^{0,0} \xrightarrow{\bar{\partial}_b} \Lambda^{0,1} \xrightarrow{\bar{\partial}_b} \cdots \xrightarrow{\bar{\partial}_b} \Lambda^{0,n} \longrightarrow 0$$

where $\Lambda^{0,j}$ is the space of j-forms of purely antiholomorphic type. If we impose a Riemannian metric on X, we can form the formal adjoint ϑ_b of $\bar{\partial}_b$ and thence the Laplacian $\Box_b = \bar{\partial}_b\vartheta_b + \vartheta_b\bar{\partial}_b$. $\Box_b$ is nonelliptic; however, according to a theorem of Kohn, for $1 \leqq j \leqq n-1$, $\Box_b$ satisfies the estimates

$$(1) \qquad \|\phi\|_{s+1} \leqq c_s(\|\Box_b\phi\|_s + \|\phi\|_0), \qquad s = 0, 1, 2, \cdots,$$

for all $\phi \in \Lambda^{0,j}$ with compact support. (Here $\| \ \|_s$ is the L^2 Sobolev norm of order s.) These estimates imply that $\Box_b$ is hypoelliptic; moreover, if X is compact, the nullspace $\mathcal{N}$ of $\Box_b$ is finite-dimensional and there is an operator G on $\Lambda^{0,j}$ satisfying

$$\|G\phi\|_{s+1} \leqq c_s \|\phi\|_s \qquad (\phi \in \Lambda^{0,j}, s = 0, 1, 2, \cdots)$$

and

$$G\Box_b = \Box_b G = I - P$$

where P is the orthogonal projection onto $\mathcal{N}$.

Kohn's method unfortunately gives no clue as to how to compute G. Our purpose here is to construct G (modulo smoothing operators) as an

---

AMS (MOS) subject classifications (1970). Primary 35B45, 35C15, 35H05, 35N15, 47G05; Secondary 32F15, 43A80, 44A25.

Key words and phrases. Tangential Cauchy-Riemann operators, subelliptic operators, regularity of solutions, fundamental solutions, integral operators, analysis on the Heisenberg group, L^p estimates, Lipschitz estimates.

explicit integral operator and to derive sharp estimates for $\bar{\partial}_b$ from this representation. Our method will be to construct an exact fundamental solution for $\Box_b$ on a particular manifold—which incidentally yields some interesting examples of hypoelliptic behavior—and then to transfer this solution to a general X.

1. **Analysis on the Heisenberg group.** Let $N \subset \boldsymbol{C}^{n+1}$ be the real hypersurface

$$N = \left\{ \zeta \in \boldsymbol{C}^{n+1} : \sum_1^n |\zeta_j|^2 = \operatorname{Im} \zeta_0 \right\}$$

N is the boundary of the generalized upper half-plane $\{\zeta : \sum_1^n |\zeta_j|^2 < \operatorname{Im} \zeta_0\}$, which is holomorphically equivalent to the unit ball in $\boldsymbol{C}^{n+1}$. We take $(x_1, \cdots, x_n, y_1, \cdots, y_n, t)$ as coordinates on N where $x_j = \operatorname{Re} \zeta_j$, $y_j = \operatorname{Im} \zeta_j$, $t = \operatorname{Re} \zeta_0$; we also write $z_j = x_j + iy_j$ and $z = (z_1, \cdots, z_n)$.

N is strongly pseudoconvex; moreover, N has a natural identification with a nilpotent Lie group (the Heisenberg group; cf. [7]). The group law is given by

$$(z, t)(z', t') = \left(z + z', t + t' + 2 \operatorname{Im} \sum_1^n z_j \bar{z}'_j \right).$$

It is easy to verify that

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}$$

form a basis for the Lie algebra of N. Also, the forms $d\bar{z}_1, \cdots, d\bar{z}_n$ are a left-invariant basis for the antiholomorphic one-forms on N.

$\bar{\partial}_b$ is a left-invariant operator on N, and it is not hard to compute it explicitly. If we set $Z_j = \frac{1}{2}(X_j - iY_j) = (\partial/\partial z_j) + i\bar{z}_j(\partial/\partial t)$, then

$$\bar{\partial}_b \left(\sum_J \phi_J \, d\bar{z}^J \right) = \sum_J \sum_{k=1}^n (\bar{Z}_k \phi_J) \, d\bar{z}_k \wedge d\bar{z}^J.$$

Here J is a multi-index and $d\bar{z}^J$ denotes a wedge product of $d\bar{z}$'s.

We impose the left-invariant metric on N which makes $Z_1, \cdots, Z_n$, $\bar{Z}_1, \cdots, \bar{Z}_n$, T orthonormal. Straightforward computation shows that the action of $\Box_b$ on $\Lambda^{0,j}$ is given by

$$\Box_b \left(\sum_J \phi_J \, d\bar{z}^J \right) = -\sum_J (\mathscr{L}_{n-2j} \phi_J) \, d\bar{z}^J$$

where, for $\alpha \in \boldsymbol{C}$,

$$\mathscr{L}_\alpha = \frac{1}{2} \sum_1^n (Z_k \bar{Z}_k + \bar{Z}_k Z_k) - i\alpha T.$$

The study of $\square_b$ is therefore reduced to the study of the left-invariant scalar operators $\mathscr{L}_\alpha$, $\alpha=n, n-2, \cdots, -n$.

We introduce the norm function $\rho(z,t)=(|z|^4+t^2)^{1/4}$ on N, which arises naturally in the study of singular integrals on N [6]. In [3] Folland showed that there is a constant $c_0\neq 0$ such that $c_0^{-1}\rho^{-2n}$ is a fundamental solution for $\mathscr{L}_0$. From homogeneity and symmetry considerations it is then natural to search for a fundamental solution for $\mathscr{L}_\alpha$ of the form $\phi_\alpha(z,t)=\rho^{-2n}(z,t)f(t/\rho^2)$. The equation $\mathscr{L}_\alpha\phi_\alpha=\delta$ (where δ is the point mass at 0) leads to an ordinary differential equation for f which can be solved explicitly, and the candidate for a fundamental solution turns out to be

$$\phi_\alpha(z,t) = (t+i\,|z|^2)^{-(n+\alpha)/2}(t-i\,|z|^2)^{-(n-\alpha)/2}.$$

THEOREM 1.

$$\mathscr{L}_\alpha\phi_\alpha = c_\alpha\delta \quad \textit{where } c_\alpha = \frac{-i^{-\alpha}2^{2-2n}\pi^{n+1}}{\Gamma(\frac{1}{2}(n+\alpha))\Gamma(\frac{1}{2}(n-\alpha))}.$$

COROLLARY. *$\mathscr{L}_\alpha$ is hypoelliptic if and only if $\pm\alpha\neq n, n+2, n+4, \cdots$.*

For, if $\pm\alpha\neq n, n+2, n+4, \cdots$, then $c_\alpha\neq 0$ and $c_\alpha^{-1}\phi_\alpha$ is a fundamental solution for $\mathscr{L}_\alpha$ which is C^∞ away from 0, whence $\mathscr{L}_\alpha$ is hypoelliptic. Otherwise, $c_\alpha=0$, so that ϕ_α is a nonsmooth solution of $\mathscr{L}_\alpha\phi_\alpha=0$.

The family of operators $\mathscr{L}_\alpha$ bears some resemblance to an example of Grušin [5] which also involves hypoellipticity of an operator for "almost all" values of a parameter.

The occurrence of the "bad values" of α can be explained in terms of the representation theory of N. According to the Stone-von Neumann theorem, for each real $\lambda\neq 0$ there is a unique irreducible representation π_λ of N on $L^2(\mathbf{R}^n)$ such that $\pi_\lambda(X_j)=-\partial/\partial\xi_j$, $\pi_\lambda(Y_j)=4i\lambda\xi_j$, $\pi_\lambda(T)=i\lambda$ where $\xi_1, \cdots, \xi_n$ are coordinates on $\mathbf{R}^n$, and $L^2(N)$ is a direct integral of these representations. (See [2].) Setting $\eta=2|\lambda|^{1/2}\xi$, we have

$$\pi_\lambda(\mathscr{L}_\alpha) = |\lambda|\sum_1^n[(\partial^2/\partial\eta_j^2)-\eta_j^2]+\lambda\alpha.$$

Thus $\pi_\lambda(\mathscr{L}_\alpha)$ is invertible for (almost) all λ if and only if $\pm\alpha$ is not an eigenvalue of the n-dimensional Hermite operator $\sum_1^n[\eta_j^2-(\partial^2/\partial\eta_j^2)]$. But these eigenvalues are well known to be $n, n+2, n+4, \cdots$.

If α is not an exceptional value, the equation $\mathscr{L}_\alpha u=f$ is solved for reasonable f by $u=f*(c_\alpha^{-1}\phi_\alpha)$, where $*$ denotes convolution on the group N. We can use this fact to derive sharp versions of the estimates (1) for $\mathscr{L}_\alpha$. If $U\subset N$ is open, $1\leqq p\leqq\infty$, $k\in\mathbf{R}$, let $L_k^p(U)$ be the L^p Sobolev space of order k on U. For $k=0, 1, 2, \cdots$, we define $S_k^p(U)$ to be the space of all

$f \in L^p_{k/2}(U)$ such that $D^\gamma f \in L^p(U)$ for all $|\gamma| \leqq k$ where

$$D = (X_1, \cdots, X_n, Y_1, \cdots, Y_n).$$

S^p_k has an obvious norm.

THEOREM 2. *Given $U \subset N$, $V \subset\subset U$, $\pm\alpha \neq n, n+2, n+4, \cdots$ and f a function on U, let u be any solution of $\mathscr{L}_\alpha u = f$ on U. If $f \in S^p_k(U)$ and $1 < p < \infty$ then $u \in S^p_{k+2}(V)$; also, if $f \in L^p(U)$, $q^{-1} = p^{-1} - (n+1)^{-1}$, and $1 < p < q < \infty$, then $u \in L^q(V)$.*

The essential point of the proof is the fact that the distribution derivatives $D^\gamma \phi_\alpha$ ($|\gamma| = 2$) and $T\phi_\alpha$ are singular integral kernels à la Knapp-Stein [**6**] (plus, perhaps, multiples of δ), and the corresponding convolutions are known to be bounded on L^p, $1 < p < \infty$ (cf. [**1**], [**7**]). The $L^p - L^q$ estimates were announced in Stein [**8**].

2. **General strongly pseudoconvex manifolds.** Let X be a strongly pseudoconvex $(2n+1)$-manifold as in §0. We choose a nonvanishing real vector field T which is complementary to the complex directions on X, so that $CTX = T_{1,0}X \oplus T_{0,1}X \oplus C \cdot T$. Replacing T by $-T$ if necessary, the Levi form $\langle\ ,\ \rangle$ on $T_{1,0}X$ given for $Z_1, Z_2 \in C^\infty(T_{1,0}X)$ by

$$[Z_1, \bar{Z}_2] = -2i\langle Z_1, Z_2\rangle T \text{ modulo } C^\infty(T_{1,0}X \oplus T_{0,1}X)$$

is positive definite. We extend $\langle\ ,\ \rangle$ to a Hermitian metric on X by requiring $T_{1,0}X \perp T_{0,1}X \perp T$ and $\langle T, T\rangle = 1$, and consider the Laplacian $\Box_b$ associated to this metric. We work locally and fix once and for all an orthonormal frame $Z_1, \cdots, Z_n$ for $T_{1,0}X$. Further we denote the dual frame for $T^*_{1,0}X$ by $\omega_1, \cdots, \omega_n$.

In this setup X looks locally like the Heisenberg group modulo small error terms, in the sense provided by the following two lemmas.

LEMMA 1. *If $\phi = \sum_J \phi_J \bar{\omega}^J \in \Lambda^{0,j}$, then*

$$\Box_b \phi = \sum_J \left[-\tfrac{1}{2} \sum (Z_k \bar{Z}_k + \bar{Z}_k Z_k) + (n - 2j) iT\right](\phi_J)\bar{\omega}^J$$

modulo terms of order one and zero not involving differentiation in the T direction.

LEMMA 2. *For each $\xi \in X$ there exist local coordinates $x^\xi_1, \cdots, x^\xi_n, y^\xi_1, \cdots, y^\xi_n, t^\xi$ on a neighborhood U_ξ of ξ, which are centered at ξ and depend smoothly on ξ, such that with $z^\xi_k = x^\xi_k + iy^\xi_k$, on U_ξ the vector fields Z_k and T take the form*

$$Z_k = \frac{\partial}{\partial z^\xi_k} + i\bar{z}^\xi_k \frac{\partial}{\partial t^\xi} + \sum \left(a_{km} \frac{\partial}{\partial z^\xi_m} + b_{km} \frac{\partial}{\partial \bar{z}^\xi_m}\right) + c_k \frac{\partial}{\partial t^\xi},$$

$$T = \frac{\partial}{\partial t^\xi} + \sum \left(\alpha_m \frac{\partial}{\partial z^\xi_m} + \beta_m \frac{\partial}{\partial \bar{z}^\xi_m}\right) + \gamma \frac{\partial}{\partial t^\xi}$$

where a_{km}, b_{km}, α_m, β_m, *and* γ *vanish to first order at* ξ, *and* c_k *vanishes to first order in* t^ξ *and to second order in* z_m^ξ *and* $\bar z_m^\xi$, $m=1, \cdots, n$.

These coordinates are constructed using exponentials of linear combinations of Z_k, $\bar Z_k$, and T. In case X is realized as a hypersurface in a complex manifold M, we can also construct them by restricting certain distinguished holomorphic coordinates on M to X.

We can now construct a parametrix for $\Box_b$ on $\Lambda^{0,j}$, $1 \leqq j \leqq n-1$. By applying a partition of unity it suffices to consider forms supported in a fixed compact set V. Let $\Omega=\{(\eta, \xi) \in X\times X: \eta \in U_\xi\}$, and choose $\psi \in C_0^\infty(\Omega)$ which$=1$ on a neighborhood of the diagonal in $V\times V$. Define the double form $K_j \in \Lambda^{0,j} \boxtimes \Lambda^{2n+1-j}$ by

$$K_j(\eta, \xi) = -c_{n-2j}^{-1}\psi(\eta, \xi)(t^\xi(\eta) + i\,|z^\xi(\eta)|^2)^{j-n} \\ \times (t^\xi(\eta) - i\,|z^\xi(\eta)|^2)^{-j} \sum_J \bar\omega^J(\eta) \otimes (*\bar\omega^J)(\xi).$$

Define the operator K on $\{\phi \in \Lambda^{0,j}: \operatorname{supp} \phi \subset V\}$ by

$$K\phi(\eta) = \int_\xi K_j(\eta, \xi) \wedge \phi(\xi),$$

and set $S=I-\Box_b K$. With the Sobolev spaces $S_k^p=S_k^p(V)$ defined as in §1, we then have

THEOREM 3. *K is bounded from S_k^p to S_{k+2}^p $(1<p<\infty)$ and from L^p to L^q $(q^{-1}=p^{-1}-(n+1)^{-1}$, $1<p<q<\infty)$. S is bounded from S_k^p to S_{k+1}^p $(1<p<\infty)$ and from L^p to L^q $(q^{-1}=p^{-1}-\frac{1}{2}(n+1)^{-1}$, $1<p<q<\infty)$.*

COROLLARY. $I-\Box_b K(\sum_0^{m-1} S^k)=S^m$ *is bounded from* S_k^p *to* S_{k+m}^p.

Thus we have a right inverse to $\Box_b$ modulo smoothing operators of arbitrarily high order. The corresponding left inverse is obtained by using the adjoint operator K^*; the analogues of Theorem 3 and its corollary hold here also. (The main point is to observe that the coordinates of Lemma 2 are essentially symmetric in ξ and η.)

It is also possible to obtain estimates for K and S in terms of the non-isotropic Lipschitz norms introduced in Stein **[8]**.

Details and proofs will appear in a later publication.

REFERENCES

1. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, 1971.

2. J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents*. II, Bull. Soc. Math. France **85** (1957), 325–388. MR **20** #1928.

3. G. B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79** (1973), 373–376.

4. G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Studies no. 75, Princeton Univ. Press, Princeton, N.J., 1972.

5. V. V. Grušin, *On a class of hypoelliptic operators*, Mat. Sb. **83** (**125**) (1970), 456–473=Math. USSR Sb. **12** (1970), 458–476. MR **43** #5158.

6. A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups*, Ann. of Math. **93** (1971), 489–578.

7. A. Korányi and S. Vági, *Singular integrals in homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa **25** (1971), 575–648.

8. E. M. Stein, *Singular integrals and estimates for the Cauchy-Riemann equations*, Bull. Amer. Math. Soc. **79** (1973), 440–445.

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

Department of Mathematics, Princeton University, Princeton, New Jersey 08540

Current address (G. B. Folland): Department of Mathematics, University of Washington, Seattle, Washington 98195