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CELL-LIKE MAPPINGS OF HILBERT CUBE MANIFOLDS: APPLICATIONS TO SIMPLE HOMOTOPY THEORY

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ABSTRACT. In this note an infinite-dimensional result is established which implies the following finite-dimensional result as a special case: If K, L are finite CW-complexes and f is a map of K onto L such that each point-inverse has trivial shape, then f is a simple homotopy equivalence.

1. **Introduction.** A *Hilbert cube manifold*, or *Q-manifold*, is a separable metric manifold modeled on the Hilbert cube Q. A mapping $f: X \to Y$ is said to be *CE*, or *cell-like*, provided that f is onto, proper (i.e. the inverse image of each compactum is compact), and each point-inverse $f^{-1}(y)$ has trivial shape (in the sense of Borsuk [**1**]). Here is the main result of this note.

THEOREM 1. *If X, Y are Q-manifolds and $f: X \to Y$ is a CE mapping, then f is proper homotopic to a homeomorphism of X onto Y.*

The key technical result needed for the proof of Theorem 1 is the solution of an infinite-dimensional *CE* handle problem, which is stated in Lemma 2 here and is the main result of [**7**]. The proof of Lemma 2 uses a considerable amount of infinite-dimensional topology along with the torus technique of [**10**], which was crucial in establishing a corresponding finite-dimensional result.

A CW-complex is *strongly locally-finite* provided that it is the union of a countable, locally-finite collection of finite subcomplexes. The following is an application of Theorem 1 to infinite simple homotopy equivalences of strongly locally-finite CW-complexes (see [**9**] for a definition of an infinite simple homotopy equivalence).

THEOREM 2. *If K, L are strongly locally-finite* CW*-complexes and f: $K \to L$ is a CE mapping, then f is an infinite simple homotopy equivalence.*

This generalizes a result of the author's [**6**], where it was shown that any homeomorphism between strongly locally-finite CW-complexes is an infinite simple homotopy equivalence. We remark that Cohen [**8**] had

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previously established a version of Theorem 2 for CE mappings of finite simplicial complexes in which the mappings are PL. More recently R. D. Edwards has proved a version of Theorem 2 for arbitrary CE mappings of countable, locally-finite, simplicial complexes.

2. **Some lemmas.** If $f, g: X \to Y$ are maps and $\mathscr{U}$ is an open cover of Y, then we say that f is *$\mathscr{U}$-close* to g provided that for each $x \in X$ there exists a $U \in \mathscr{U}$ containing both $f(x)$ and $g(x)$. If $A \subset Y$, then we say that $f = g$ over A provided that $f^{-1}(A) = g^{-1}(A)$ and $f \mid f^{-1}(A) = g \mid g^{-1}(A)$. We also say that f is 1-1 over A provided that $f \mid f^{-1}(A)$ is 1-1.

If X is a Q-manifold, then a closed subset A of X is a *Z-set* in X provided that given any nonnull and contractible open subset U of X, $U \setminus A$ is also nonnull and contractible. The following result is established in [7].

LEMMA 1. *Let X and Y be Q-manifolds, $f: X \to Y$ be a CE mapping, and let $A \subset Y$ be a Z-set in Y. If $\mathscr{U}$ is an open cover of Y, then there exists a CE mapping $g: X \to Y$ such that g is* 1-1 *over A, $g^{-1}(A)$ is a Z-set in X, and g is $\mathscr{U}$-close to f.*

For notation for the next result let R^n denote Euclidean n-space (where $R^1 = R$) and let B^n_r denote the standard n-ball of radius r, with interior $\mathrm{Int}(B^n_r)$ and boundary $\mathrm{Bd}(B^n_r) = S^{n-1}_r$. The following is the CE handle result of [7].

LEMMA 2. *Let X be a Q-manifold and let $f: X \to B^k_1 \times R^n \times Q$ be a CE mapping, for $k \geqq 0$ and $n \geqq 1$, such that f is* 1-1 *over $S^{k-1}_1 \times R^n \times Q$ and $f^{-1}(S^{k-1}_1 \times R^n \times Q)$ is a Z-set in X. Then there exists a CE mapping $g: X \to B^k_1 \times R^n \times Q$ such that $g = f$ over $(S^{k-1}_1 \times R^n \times Q) \cup (B^k_1 \times (R^n \setminus \mathrm{Int}(B^n_2)) \times Q)$ and g is* 1-1 *over $B^k_1 \times B^n_1 \times Q$.*

We now use Lemmas 1 and 2 to prove the following result which will be needed in the proof of Theorem 1.

LEMMA 3. *Let X and Y be Q-manifolds, $f: X \to Y$ be a CE mapping, K be a finite simplicial complex, and let $\varphi: K \times Q \times R \to Y$ be an open embedding. Then there exists CE mapping $g: X \to Y$ such that $g = f$ over $Y \setminus \varphi(K \times Q \times (-2, 2))$, g is* 1-1 *over $\varphi(K \times Q \times [-1, 1])$, and*

$$g \simeq f \text{ rel } X \setminus f^{-1}\varphi(K \times Q \times (-2, 2)).$$

PROOF. By taking a regular neighborhood of K in an Euclidean space we get a compact, combinatorial, n-manifold M which is simple homotopy equivalent to K. The main result of [11] asserts that if A and B are simple homotopy equivalent finite simplicial complexes, then $A \times Q$ is homeomorphic to $B \times Q$. Thus no generality is lost by replacing K with a compact, combinatorial, n-manifold M.

By choosing a PL handle decomposition of M we can write $M = M_{-1} \cup M_0 \cup \cdots \cup M_n$, where M_{-1} is a regular neighborhood of ∂M and each M_i is a compact PL submanifold of M which is obtained from M_{i-1} by adding handles of index i. For each i, $-1 \leqq i \leqq n$, we will show how to construct a CE mapping $g_i : X \to Y$ such that $g_i = f$ over $Y \backslash \varphi(M \times Q \times (-2, 2))$, g_i is 1-1 over a neighborhood of

$$\varphi(M_i \times Q \times [-1, 1]),$$

and $g_i \simeq f \operatorname{rel} X \backslash f^{-1}\varphi(M \times Q \times (-2, 2))$.

For the construction of g_{-1} we first note that $\varphi(\partial M \times Q \times [-1, 1])$ is a Z-set in the Q-manifold $\varphi(M \times Q \times (-2, 2))$. Applying Lemma 1 to the restricted CE mapping

$$f| : f^{-1}\varphi(M \times Q \times (-2, 2)) \to \varphi(M \times Q \times (-2, 2))$$

we can find a CE mapping $g'_{-1} : f^{-1}\varphi(M \times Q \times (-2, 2)) \to \varphi(M \times Q \times (-2, 2))$ such that g'_{-1} is 1-1 over $\varphi(\partial M \times Q \times [-1, 1])$, $(g'_{-1})^{-1}\varphi(\partial M \times Q \times [-1, 1])$ is a Z-set in $f^{-1}\varphi(M \times Q \times (-2, 2))$, and g'_{-1} is $\mathscr{U}$-close to $f|$, for any prechosen open cover $\mathscr{U}$ of $\varphi(M \times Q \times (-2, 2))$. By choosing $\mathscr{U}$ sufficiently fine we can extend g'_{-1} to a CE mapping $\tilde{g}_{-1} : X \to Y$ such that $\tilde{g}_{-1} = f$ over $Y \backslash \varphi(M \times Q \times (-2, 2))$ and $\tilde{g}_{-1} \simeq f \operatorname{rel} X \backslash f^{-1}\varphi(M \times Q \times (-2, 2))$. The collaring theorem of [3], which asserts that every Q-manifold which is a Z-set in another Q-manifold is also collared in that Q-manifold, implies that $\tilde{g}_{-1}^{-1}\varphi(\partial M \times Q \times [-1, 1])$ has a collar neighborhood in X. That is, there exists an open embedding

$$\alpha : \tilde{g}_{-1}^{-1}\varphi(\partial M \times Q \times [-1, 1]) \times [0, 1) \to X$$

such that $\alpha(x, 0) = x$, for all $x \in \tilde{g}_{-1}^{-1}\varphi(\partial M \times Q \times [-1, 1])$. It is also true that $\partial M \times [-1, 1]$ has a collar neighborhood in $M \times (-2, 2)$ which contains $M_{-1} \times [-1, 1]$. This is a finite-dimensional problem and uses the fact that M_{-1} is a regular neighborhood of ∂M. Thus $\varphi(\partial M \times Q \times [-1, 1])$ has a collar neighborhood in $\varphi(M \times Q \times (-2, 2))$ which contains $\varphi(M_{-1} \times Q \times [-1, 1])$. Using these collar neighborhoods it is easy to modify $\tilde{g}_{-1}$ to get our desired g_{-1}.

To construct g_i, $0 \leqq i \leqq n$, we just inductively work our way through the handles of the decomposition, applying Lemma 2 repeatedly. We leave the details to the reader.

Finally we will need a relative version of Theorem 1 for the compact case.

LEMMA 4. *Let X, Y be compact Q-manifolds, $A \subset Y$ be a Z-set, and let $f : X \to Y$ be a CE mapping such that f is* 1-1 *over A and $f^{-1}(A)$ is a Z-set in X. Then there exists a homeomorphism $g : X \to Y$ such that $g = f$ on $f^{-1}(A)$ and $g \simeq f \operatorname{rel} f^{-1}(A)$.*

PROOF. The triangulation theorem of [4] asserts that Y is homeomorphic to $K \times Q$, for some finite simplicial complex K. The comments made at the beginning of the proof of Lemma 3 imply that Y is homeomorphic to $M \times Q$, for some compact, combinatorial, n-manifold M. Thus Y can be replaced by $M \times Q$ and the main theorem of [2] concerning Z-sets in Q-manifolds implies that we can assume $A \subset \partial M \times Q$. [The main theorem of [3] implies that there is a homeomorphism h of $M \times Q$ onto $M \times [0, 1] \times Q$ which takes A into $M \times \{0\} \times Q$. Then $h(A) \subset \partial(M') \times Q$, where $M' = M \times [0, 1]$.]

Using Lemma 1 let $\tilde{g}_{-1}: X \to M \times Q$ be a CE mapping such that $\tilde{g}_{-1}$ is 1-1 over $\partial M \times Q$, $\tilde{g}_{-1}^{-1}(\partial M \times Q)$ is a Z-set in X, $\tilde{g}_{-1}$ is $\mathscr{U}$-close to f, for any prechosen open cover $\mathscr{U}$ of Y, and $\tilde{g}_{-1} = f$ over A. [To see this we apply Lemma 1 to the restricted CE mapping $f|: X \backslash f^{-1}(A) \to (M \times Q) \backslash A$.] If $\mathscr{U}$ is sufficiently fine, then we have $\tilde{g}_{-1} \simeq f \text{ rel } f^{-1}(A)$.

As in the proof of Lemma 3 let $M = M_{-1} \cup M_0 \cup \cdots \cup M_n$ be a PL handle decomposition of M and modify $\tilde{g}_{-1}$ to get a CE mapping $g_{-1}: X \to Y$ such that g_{-1} is 1-1 over a neighborhood of $M_{-1} \times Q$, $g_{-1} = \tilde{g}_{-1}$ over $\partial M \times Q$, and $g_{-1} \simeq \tilde{g}_{-1} \text{ rel } \tilde{g}_{-1}^{-1}(\partial M \times Q)$. Then we inductively work our way through the handles in a standard manner, applying Lemma 2 at each step.

3. **Proof of Theorem 1.** In Lemma 4 we treated the compact case so let us assume that we have a CE mapping $f: X \to Y$, where X and Y are noncompact Q-manifolds. The triangulation theorem of [5] implies that we can replace Y by $K \times Q$, where K is a countable, locally-finite, simplicial complex. Write $K = \bigcup_{n=1}^{\infty} K_n$, where each K_n is a finite subcomplex of K such that $K_n \subset \operatorname{Int}(K_{n+1})$ and $\operatorname{Bd}(K_n)$ is a finite subcomplex which is PL bicollared in K. [To achieve this we might have to subdivide K.] Thus for each $n \geqq 1$ we have an open embedding

$$\varphi_n: \operatorname{Bd}(K_n) \times Q \times R \to (\operatorname{Int}(K_{n+1}) \backslash K_{n-1}) \times Q$$

such that $\varphi_n(x, q, 0) = (x, q)$, for all $(x, q) \in \operatorname{Bd}(K_n) \times Q$. Moreover the φ_n's can be chosen so that their images are pairwise disjoint.

It follows from Lemma 3 that there exists a CE mapping $g_1: X \to Y$ such that

$$g_1 = f \quad \text{over } (K \times Q) \Big\backslash \bigcup_{n=1}^{\infty} \varphi_n(\operatorname{Bd}(K_n) \times Q \times (-2, 2)),$$

$$g_1 \text{ is 1-1} \quad \text{over } \bigcup_{n=1}^{\infty} \varphi_n(\operatorname{Bd}(K_n) \times Q \times [-1, 1]),$$

and g_1 is proper homotopic to f. Now consider the restricted CE mapping

$g_1|:X_1 \to Y_1$, where

$$Y_1 = (K \times Q) \Big\backslash \bigcup_{n=1}^{\infty} \varphi_n(\mathrm{Bd}(K_n) \times Q \times (-\tfrac{1}{2}, \tfrac{1}{2})) \quad \text{and} \quad X_1 = g_1^{-1}(Y_1).$$

Note that Y_1 is the union of compact Q-manifolds which are pairwise disjoint. Moreover the topological boundaries of these compact Q-manifolds (i.e. boundaries in $K \times Q$) are Z-sets in Y_1 such that $g_1|$ is 1-1 over each one and the inverse image of each one under $g_1|$ is a Z-set in X_1. Thus Lemma 4, applied to these compact Q-manifolds, gives a homeomorphism $\tilde{g}_2:X_1 \to Y_1$ which extends to a homeomorphism $g_2:X \to Y$ such that $g_2 = g_1$ over $\bigcup_{n=1}^{\infty} \varphi_n(\mathrm{Bd}(K_n) \times Q \times [-\frac{1}{2}, \frac{1}{2}])$ and g_2 is proper homotopic to g_1. Thus g_2 is our required homeomorphism.

4. **Proof of Theorem 2.** In [6] it was shown that if K, L are strongly locally-finite CW-complexes and $f:K \to L$ is a proper homotopy equivalence, then f is an infinite simple homotopy equivalence if $f \times \mathrm{id}:K \times Q \to L \times Q$ is proper homotopic to a homeomorphism of $K \times Q$ onto $L \times Q$.

If $f:K \to L$ is a CE mapping, then $f \times \mathrm{id}:K \times Q \to L \times Q$ is also a CE mapping. It follows from [12] that $K \times Q$ and $L \times Q$ are Q-manifolds. Thus Theorem 2 follows from Theorem 1.

5. **Open questions.** We list here two questions which are related to Theorem 2 but which do not appear to be susceptible to the same techniques. In what follows let A be a compact ANR and let K, L be finite CW-complexes.

Question 1. *If $f:K \to A$, $g:L \to A$ are CE mappings, then does there exist a simple homotopy equivalence $h:K \to L$ such that $gh \simeq f$?*

Question 2. *If $f:A \to K$, $g:A \to L$ are CE mappings, then does there exist a simple homotopy equivalence $h:K \to L$ such that $hf \simeq g$?*

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