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THE ITERATIVE SOLUTION OF THE EQUATION $y \in x + Tx$ FOR A MONOTONE OPERATOR T IN HILBERT SPACE[1]

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ABSTRACT. Suppose T is a multivalued monotone operator with open domain $D(T)$ in a Hilbert space and $y \in R(I + T)$. Then there exist a neighborhood $N \subset D(T)$ of $\bar{x} = (I + T)^{-1}y$ and a real number $\sigma_1 > 0$ such that for any $\sigma \geqq \sigma_1$, any initial guess $x_1 \in N$, and any single-valued section T_0 of T, the sequence generated from x_1 by

$$x_{n+1} = x_n - (n + \sigma)^{-1}(x_n + T_0 x_n - y)$$

remains in $D(T)$ and converges to $\bar{x}$ with estimate $\|x_n - \bar{x}\| = O(n^{-1/2})$. The sequence $\{x_n + T_0 x_n\}$ converges $(C, 1)$ to y. No continuity assumptions of any kind are imposed on T_0.

If H is a real or complex Hilbert space with inner product $(\cdot, \cdot)$, a *multivalued monotone operator* on H is a subset T of $H \times H$ for which $\operatorname{Re}(u - v, x - y) \geqq 0$ whenever $[x, u]$, $[y, v] \in T$. We write Tx for $\{y \in H : [x, y] \in T\}$, $D(T) = \{x : Tx \neq \emptyset\}$ (the *effective domain* of T), $T(A) = \bigcup \{Tx : x \in A\}$ if $A \subset H$, and $R(T) = T(H)$. I denotes the identity operator on H, so $I + T = \{[x, x + y] : [x, y] \in T\}$ and $(I + T)^{-1} = \{[x + y, x] : [x, y] \in T\}$. T is *locally bounded* at x if there exists a neighborhood N of x (in the norm topology) for which $T(N)$ is bounded. A single-valued section of T is a subset T_0 of T for which $T_0 x$ is a singleton set for each x in $D(T)$; we follow the traditional abuse of terminology and refer to $T_0 x$ as the *element* in the singleton set.

One of the earliest problems in the theory of monotone operators was to solve the equation $y \in x + Tx$ for x, given an element y of H and a monotone operator T. The initial existence theorems (Vaĭnberg [**10**], Zarantonello [**12**]) were constructive in nature, but assumed that the operator T was single-valued and Lipschitzian; later existence results (Minty [**7**], Browder [**1**]) were proven under unusually weak continuity assumptions on T, but were nonconstructive in nature. Subsequent iteration methods have weakened the Lipschitz assumptions on T (Petryshn [**8**], Zarantonello [**11**]).

In this note we return to the iterative techniques, with the difference that we make *no* assumptions of continuity. Supposing that the equation $y \in x + Tx$ has a solution x, we calculate that solution as the limit of an iteratively constructed sequence with an explicit error estimate. Naturally,

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the sequence of iterates converges more slowly than in the Lipschitzian case—the error estimate is $O(n^{-1/2})$ instead of $O(c^n)$ for some $0 < c < 1$.

THEOREM. *Suppose T is a multivalued monotone operator with open domain $D(T)$ in a Hilbert space H and $y \in R(I + T)$. Then there exist a neighborhood $N \subset D(T)$ of $\bar{x} = (I + T)^{-1}y$ and a real number $\sigma_1 > 0$ such that for any $\sigma \geqq \sigma_1$, any initial guess $x_1 \in N$, and any single-valued section T_0 of T, the sequence $\{x_n\}$ generated from x_1 by*

$$x_{n+1} = x_n - (n + \sigma)^{-1}(x_n + T_0 x_n - y) \tag{1}$$

remains in $D(T)$ and converges to $\bar{x}$ with estimate $\|x_n - \bar{x}\| = O(n^{-1/2})$. The sequence $\{x_n + T_0 x_n\}$ converges $(C, 1)$ to y.

REMARK. The iteration (1) is a normal Mann process in the sense of Dotson [**3**] (see also Mann [**6**]).

PROOF OF THEOREM. It is well known that $(I + T)^{-1}$ is a single-valued mapping, so $\bar{x} = (I + T)^{-1}y$ is uniquely defined. Choose $\bar{u} \in T\bar{x}$ such that $y = \bar{x} + \bar{u}$. As an aside we note that if already $y \in R(I + T_0)$ then $\bar{u} = T_0\bar{x}$ because $(I + T_0)^{-1} \subset (I + T)^{-1}$.

A monotone operator is locally bounded at each interior point of its effective domain (Browder [**2**], Rockafellar [**9**]); thus T is locally bounded at $\bar{x}$, and we may choose $N = B_d(\bar{x})$, the closed ball of radius $d > 0$ centered at $\bar{x}$, in such a way that $N \subset D(T)$ and $T(N)$ is bounded. Put $\sigma_1 = [\operatorname{diam} T(N)/d]^2$. Then $\sigma_1 > 0$ and

$$\operatorname{diam} T(N) \leqq d\sigma^{1/2} \text{ if } \quad \sigma \geqq \sigma_1. \tag{2}$$

Put $t_n = (n + \sigma)^{-1}$, $d_n = (n + \sigma - 1)^{-1/2}$. Starting with an initial guess $x_1 \in N$ and any single-valued section T_0 of T, define a sequence $\{x_n\}$ inductively by (1). That $\{x_n\}$ can in fact be continued indefinitely follows from

(3) for all $n \geqq 1$, x_n is well defined and $\|x_n - \bar{x}\| \leqq d_n\, d\sigma^{1/2}$.

We prove (3) by induction on n.

For $n = 1$, x_n is certainly well defined and $\|x_n - \bar{x}\| \leqq d_1\, d\sigma^{1/2} = d$ because x_1 was originally chosen from $B_d(\bar{x}) = N$. Next suppose that (3) has been proven for a particular value of n. Then

$$\|x_n - \bar{x}\| \leqq d_n\, d\sigma^{1/2} \leqq d_1\, d\sigma^{1/2} = d$$

so $x_n \in N \subset D(T) = D(T_0)$; thus x_{n+1} is well defined by (1). Since $y = \bar{x} + \bar{u}$, (1) also implies

$$x_{n+1} - \bar{x} = (1 - t_n)(x_n - \bar{x}) - t_n(T_0 x_n - \bar{u})$$

so

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= (1 - t_n)^2 \|x_n - \bar{x}\|^2 - 2t_n(1 - t_n) \\ &\quad \cdot \operatorname{Re}(T_0x_n - \bar{u}, x_n - \bar{x}) + t_n^2 \|T_0x_n - \bar{u}\|^2 \\ &\leqq (1 - t_n)^2 \|x_n - \bar{x}\|^2 + t_n^2 \|T_0x_n - \bar{u}\|^2 \end{aligned} \tag{4}$$

since $T_0x_n \in Tx_n$, $\bar{u} \in T\bar{x}$, T is monotone, and $t_n(1 - t_n) > 0$. But we also have $\|T_0x_n - \bar{u}\| \leqq \operatorname{diam} T(N)$ since T_0x_n and $\bar{u}$ belong to $T(N)$, so when (2) and (4) are combined with the induction hypothesis $\|x_n - \bar{x}\| \leqq d_n\, d\sigma^{1/2}$ there results

$$\|x_{n+1} - \bar{x}\|^2 \leqq [(1 - t_n)^2 d_n^2 + t_n^2]\, d^2\sigma = d_{n+1}^2\, d^2\sigma,$$

so that $\|x_{n+1} - \bar{x}\| \leqq d_{n+1}\, d\sigma^{1/2}$. This completes the proof of (3) by induction. Since $d_n = O(n^{-1/2})$ we have also established the error estimate of the Theorem.

As for the $(C, 1)$ convergence of $\{x_n + T_0x_n\}$ to y, we readily establish by induction the identity

$$x_{n+1} = \frac{\sigma}{n + \sigma} \cdot x_1 - \frac{1}{n + \sigma} \cdot \sum_{i=1}^{n} (T_0x_i - y),$$

so that $n^{-1} \cdot \Sigma_{i=1}^{n}\, T_0x_i = y - x_{n+1} + \sigma n^{-1}(x_1 - x_{n+1})$. It follows that

$$(C, 1) \lim_{n \to \infty} T_0x_n = y - \bar{x}.$$

Since $x_n \to \bar{x}$ strongly, also

$$(C, 1) \lim_{n \to \infty} x_n = \bar{x},$$

so finally $(C, 1) \lim_{n \to \infty} (x_n + T_0x_n) = y$. Q.E.D.

COROLLARY. *Suppose T is a continuous single-valued monotone operator with open domain $D(T)$ in a Hilbert space H and $y \in R(I + T)$. Then there exists a neighborhood $N \subset D(T)$ of $\bar{x} = (I + T)^{-1}y$ such that for any initial guess $x_1 \in N$ the sequence generated from x_1 by*

$$x_{n+1} = \frac{n}{n + 1} x_n - \frac{1}{n + 1} (Tx_n - y) \tag{5}$$

remains in $D(T)$ and converges to $\bar{x}$ with estimate $\|x_n - \bar{x}\| = O(n^{-1/2})$.

REMARK. This iteration has been considered for different families of mappings by Johnson [**5**] and Franks and Marzec [**4**].

PROOF OF COROLLARY. By the Theorem there exist a neighborhood N_1 of $\bar{x}$ and a positive integer σ such that the sequence $\{z_n\}$ generated from an initial guess $z_1 \in N_1$ by

$$z_{n+1} = z_n - (n + \sigma)^{-1}(z_n + Tz_n - y) \tag{6}$$

converges to $\bar{x}$ with error estimate $O(n^{-1/2})$. The continuity of T at $\bar{x}$ guarantees the existence of a neighborhood N of $\bar{x}$ such that for any initial guess $x_1 \in N$, the first σ terms of the sequence $\{x_n\}$ defined by (5) remain in N_1. Taking $z_1 = x_\sigma \in N_1$ and generating $\{z_n\}$ by (6), we have $x_n = z_{n-\sigma+1}$ for $n \geqq \sigma$. Thus $\{x_n\}$ remains in $D(T)$ and $x_n \to \bar{x}$ with error estimate $O((n - \sigma + 1)^{-1/2}) = O(n^{-1/2})$. Q.E.D.

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