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A FATOU THEOREM FOR THE GENERAL ONE-DIMENSIONAL PARABOLIC EQUATION[1]

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1. **Introduction.** Let R be a finite or infinite one-dimensional open interval. Our main purpose here is to characterize all positive weak solutions of the equation

$$(1.1)\qquad \partial u/\partial t = \mathscr{D}u = a(x)u_{xx} + b(x)u_x + c(x)u \quad \text{in } R \times (0, T)$$

where $T \leqq \infty$. Here $a(x) \geqq 0$, $c(x) \leqq M$,[2] $1/a(x)$, $b(x)/a(x)$ and $c(x)/a(x)$ are locally integrable in R, but otherwise the coefficients are unrestricted. The results below extend characterizations of Widder ([**17**], [**18**]) for positive solutions of the heat equation (see §1.1). In particular, we find that all positive solutions of (1.1) are of the form

$$(1.2)\qquad u(x, t) = \int_R p(t, x, y)F(dy),$$

where $p(t, x, y)$ is the fundamental solution of (1.1), if and only if the Green's function of (1.1) is not of trace class at either endpoint of R. While these results are one-dimensional, they do have the advantage that they are complete, and suggest possible generalizations in higher dimensions. Proofs will appear elsewhere.

Equation (1.1) can always be transformed into a similar equation with $c(x) = 0$; assume for the moment $c = 0$ in (1.1). Then by a simple change of variables we can write (1.1) in "Feller form"

$$(1.3)\qquad \partial u/\partial t = \mathscr{D}u = (d/dm(x))(du/ds(x)) \quad \text{in } R \times (0, T)$$

where $m(x)$ and $s(x)$ are increasing; we can also consider equation (1.3) for an arbitrary strictly increasing continuous function $s(x)$ (a "scale") and Borel measure $m(dx)$ which is positive on open sets in R. Then $\mathscr{D}$ becomes

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[2] The condition $c(x) \leqq M$ can be removed if there exists some $g(x) > 0$ in R such that $\mathscr{D}g = \lambda g$ for some λ.

the infinitesimal generator of the general nonsingular diffusion process r_t in R which terminates at the endpoints ([**3**], [**5**], [**10**], [**12**]). Indeed, in terms of r_t, $u(x, t)$ turns out to be a weak solution of (1.3) iff

(1.4) $u(r_t, T - t)$ is a local martingale, $0 < t < T$.

In this context, the problem is to determine when all local martingales of the form (1.4) are actually given by (1.2), and to classify the ones that are not. Such representation theorems have had applications in proving theorems of iterated logarithm type (see §1.1 and [**13**]–[**15**]). Particularly we would like to thank Professor Siegmund for access to the manuscript [**15**] and for many helpful conversations.

The general positive solution of (1.3) will depend on the behavior of $\mathscr{D}$ near the endpoints; we use $s(x)$ and $m(dx)$ to classify each endpoint of R into one of four types (see §2). Since the form of (1.1) or (1.3) is invariant under diffeomorphisms of R, it is sufficient to assume $R = (0, 1)$.

We remark [**10**, §4.11] that equation (1.3) (with zero boundary conditions where appropriate) always has a symmetric fundamental solution $p(t, x, y)$ with respect to the measure $m(dx)$. Then

THEOREM 1. *Let $u(x, t)$ be a nonnegative weak solution (see §2 for exact definitions) of* (1.3) *in $R \times (0, T)$. Then, there exist measures $c(ds) \geqq 0$, $\tilde{c}(ds) \geqq 0$ on $[0, T)$ and $F(dy) \geqq 0$ on R such that*

$$u(x, t) = \int_{0-}^{t} q_0(t - s, x)c(ds) + \int_{0-}^{t} q_1(t - s, x)\tilde{c}(ds) + \int_R p(t, x, y)F(dy) \tag{1.5}$$

where:

(i) *If* 0 *is an* ***accessible boundary*** (*see* §2),

$$q_0(t, x) = \lim_{a \to 0} \frac{p(t, a, x)}{s(a) - s(0)} = \frac{dp}{ds}(t, 0, x),$$

the limit existing weakly in t (i.e., the indefinite integrals converge). Similarly, if 1 *is accessible*, $q_1(t, x) = -(dp/ds)(t, 1, x)$.

(ii) *If* 0 *is an* ***entrance boundary***, $q_0(t, x) = p(t, 0+, x)$, *the limit existing as in* (i), *and* $q_1(t, x) = p(t, 1-, x)$ *if* 1 *is entrance*.

(iii) *In the remaining cases, i.e. if* 0 *is an* ***inaccessible exit*** (*I.E.*) *or* ***natural boundary***, *no term for* $q_0(t, x)$ *appears in* (1.5), *and the kernel*

$q_0(t, x)$ *of* (i) (*respectively* (ii) *if* 0 *is natural*) *exists and is identically zero.*

Thus all nonnegative solutions of (1.3) are of the form (1.2) iff both boundaries are natural or I.E. The boundary measures in (1.5) (when they exist) can be recovered from $u(x, t)$ by weak limits; in particular, if these measures have bounded densities, $u(x, t)$ itself is bounded iff there is no contribution in (1.5) from an entrance boundary.

Now, consider (1.1) for arbitrary $c(x) \leqq 0$, or, more generally,

$$\frac{\partial}{\partial t} u = \mathscr{D}_1 u = \frac{d}{dm}\left(\frac{du}{ds} - \int^x u(y)k(dy)\right) \tag{1.6}$$

where $k(dx)$ is a nonnegative Borel measure in R (see [**10**, Chapter 4]). Thus $\mathscr{D}_1 u = \mathscr{D}u - c(x)u$ if $k(dx) = c(x)m(dx)$. Let $p(t, x, y)$ be the fundamental solution of (1.6) with respect to $m(dx)$ as before, and let $h_+(x)\uparrow$, $h_-(x)\downarrow$ be two linearly independent positive solutions of

$$\mathscr{D}_1 h_+(x) = \mathscr{D}_1 h_-(x) = 0. \tag{1.7}$$

Such functions can be defined whenever $k(R) > 0$. Then

THEOREM 2. *Let $u(x, t)$ be a nonnegative weak solution in $R \times (0, T)$ of equation* (1.6). *Then, $u(x, t)$ has a representation of the form* (1.5), *where* $q_0(t, x) \not\equiv 0$ *iff*

$$\int_0^{1/2} h_+(x)h_-(x)\, m(dx) < \infty. \tag{1.8}$$

If (1.8) *converges,* $q_0(t, x) = \lim_{a\to 0} p(t, a, x)/h_+(a)$ *as in Theorem* 1, *and* $\int_0^t u(x, s)\, ds = O(h_-(x))$ *for small x for any solution $u \geqq 0$. The integral* (1.8) *always converges if* 0 *is accessible or entrance for $\mathscr{D}$. The same results hold at* 1 *mutatis mutandum.*

EXAMPLES. (1) If $\mathscr{D}_1 u = u_{xx} - (12/x^2)u$ in $(0, \infty)$, then $h_+(x) = x^4$, $h_-(x) = 1/x^3$, $m(dx) = dx$, and (1.8) converges at 0. Thus $q_0(t, x) = \lim_{a\to 0} p(t, a, x)/a^4$, and $\int_0^t u(x, s)\, ds = O(1/x^3)$ for a general positive solution of (1.6).

(2) Let $\mathscr{D}_1 u = u_{xx} - x^a u$ in $(0, \infty)$. Then ∞ is a natural boundary for $\mathscr{D}u = u_{xx}$, but (1.8) converges at ∞ for all $a > 2$. Thus, if $a > 2$, the general positive solution of (1.6) has a boundary term at ∞, and the Green's function of (1.6) is a trace class operator in $L^2(R, dx)$. Incidentally, $\mathscr{D}_1$ is of limit point type at ∞ in the sense of Weyl for all $a > 0$.

REMARK. If $k(dx) = c(x)m(dx)$, where $0 \leqq c(x) \leqq M$, the integral (1.8) converges iff 0 is accessible or entrance for $\mathscr{D}$.

1.1 The first results of this type were due to Widder [**17**], [**18**] and

Hartman and Wintner [8], who obtained Theorem 1 for positive solutions of the heat equation $u_t = u_{xx}$ for $R = (-\infty, \infty)$, $(0, \infty)$, and (a, b) for finite a, b. For this equation, the boundaries $\pm\infty$ are natural and finite boundaries are accessible. Later authors ([1], [2], [6], [16]) generalized the heat equation to (1.1) with various conditions on $a(x)$, $b(x)$ and $c(x)$ (often in higher dimensions as well) but always under conditions that exclude boundary terms at $\pm\infty$. In a different vein is a recent paper of Robbins and Siegmund [15], who use probabilistic arguments to find all positive solutions of

$$\partial u/\partial t = u_{xx} + 2u_x/x \quad \text{in } (0, \infty) \times (0, T). \tag{1.9}$$

Here ∞ is an I.E., but now 0 is an entrance boundary. This appears to be the first example of a Fatou representation (1.5) with a term from an inaccessible boundary, although the possibility is suggested by earlier work of Feller [4] and Hille [9]. Robbins and Siegmund were interested in positive solutions of (1.9) which are constant along certain curves $x = x(t)$; these can be interpreted as the probability that three-dimensional Brownian motion ever escapes (or alternately is overtaken by) an expanding sphere of radius $r = x(t)$. The unbounded term in (1.5) yields a law of iterated logarithm type for the minimum modulus of three-dimensional Brownian motion. Using Theorem 1, these arguments can be extended to Brownian motion in dimensions other than three.

The comparison of Widder's results and the general solution of (1.9) led to the conjecture of Theorem 1. In the special case $u(x, t) = e^{\lambda t}g(x)$ (where $\mathscr{D}g = \lambda g$), this was verified by Tz. L. Lai [11] by methods different from ours. He also obtained Theorem 1 under various specific growth conditions at the boundaries.

2. **Definitions.** For closed subintervals $[a, b]$ of R, set

$$g_{ab}(x, y) = \int_0^\infty p_{ab}(t, x, y)\, dt, \tag{2.1}$$

where $p_{ab}(t, x, y)$ is the fundamental solution of (1.3) or (1.6) in (a, b) with zero boundary conditions. We say that $u(x, t)$ is a *weak solution* of (1.3) in $R \times (0, T)$, if $u(x, t)$ is locally integrable in $R \times (0, T)$ and satisfies

$$\begin{aligned} \int_{t_1}^{t_2} u(x, \theta)\, d\theta = & -\int_a^b g_{ab}(x, y)(u(y, t_2) - u(y, t_1))\, m(dy) \\ & + \frac{s(b) - s(x)}{s(b) - s(a)} \int_{t_1}^{t_2} u(a, \theta)\, d\theta \\ & + \frac{s(x) - s(a)}{s(b) - s(a)} \int_{t_1}^{t_2} u(b, \theta)\, d\theta \end{aligned} \tag{2.2}$$

for a.e. a, x, b in R with $a < x < b$, and a.e. t_1, t_2 in $(0, T)$. To define a weak solution of (1.6), we replace the coefficients of the last two integrals in (2.2) by appropriate linear combinations of the solutions $h_{\pm}(x)$ of (1.7). Using a result of Lai [11], one can show that all weak solutions are actually continuous after modification on a null set in $R \times (0, T)$.

If $|s(0)| < \infty$ in (1.3), i.e. if $s(x)$ is bounded from below in R, we say that 0 is an *accessible boundary* if $I = \int_0^{1/2}(s(x) - s(0))\, m(dx) < \infty$ and is an *inaccessible exit* (I.E.) if $I = \infty$. If $s(0) = -\infty$, 0 is *entrance* if $J = \int_0^{1/2} |s(x)|\, m(dx) < \infty$ and a *natural boundary* if $J = \infty$. See [3], [10], or [12] for the meaning of these terms in terms of the behavior of r_t. In particular, 0 is accessible iff r_t can reach 0 in finite time with positive probability. An entrance boundary is inaccessible, although r_t can enter R at 0. If 0 is accessible and $T_0 = \inf\{t : r_t = 0\}$, (1.5) for $u(x, t) = \mathrm{Prob}(T_0 \leqq t \mid r_0 = x)$ shows that $q_0(t, x) = \mathrm{Prob}(T_0 \in dt)/dt$.

References

1. D. G. Aronson, *Uniqueness of positive weak solutions of second order parabolic equations*, Ann. Polon. Math. **16** (1965), 285–303. MR **31** #506.

2. D. G. Aronson and P. Besala, *Uniqueness of positive solutions of parabolic equations with unbounded coefficients*, Colloq. Math. **18** (1967), 125–135. MR **36** #2971.

3. L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., 1968, Chap. 16. MR **37** #4841.

4. W. Feller, *The parabolic differential equations and the associated semi-group of transformations*, Ann. of Math. (2) **55** (1952), 468–519. MR **13,** 948.

5. ———, *On second order differential operators*, Ann. of Math. (2) **61** (1955), 90–105. MR **16,** 824.

6. A. Friedman, *On the uniqueness of the Cauchy problem for parabolic equations*, Amer. J. Math. **81** (1959), 503–511. MR **21** #3657.

7. ———, *Uniqueness for the Cauchy problem for degenerate parabolic equations*, Pacific J. Math. (to appear).

8. P. Hartman and A. Wintner, *On the solutions of the equation of heat conduction*, Amer. J. Math. **72** (1950), 367–395. MR **12,** 104.

9. E. Hille, *Les probabilités continues en chaînes*, C. R. Acad. Sci. Paris **230** (1950), 34–35. MR **11,** 256.

10. K. Ito and H. P. McKean, Jr., *Diffusion processes and their sample paths*, Die Grundlehren der math. Wissenchaften, Band 125, Academic Press, New York; Springer-Verlag, Berlin and New York, 1965. MR **33** #8031.

11. Tz. L. Lai, *Space-time processes, parabolic functions and one-dimensional diffusions*, Trans. Amer. Math. Soc. **175** (1973), 409–438.

12. P. Mandl, *Analytical treatment of one-dimensional Markov processes*, Die Grundlehren der math. Wissenschaften, Band 151, Academia, Prague; Springer-Verlag, New York, 1968. MR **40** #930.

13. H. Robbins and D. Siegmund, *Boundary crossing probabilities for the Wiener process and sample sums*, Ann. Math. Statist. **41** (1970), 1410–1429. MR **43** #2796.

14. ———, *On the law of the iterated logarithm for maxima and minima*, Proc. VI Berkeley Sympos. (1970/71), vol. 3, pp. 51–70.

15. H. Robbins and D. Siegmund, *Statistical tests of power one and the integral representation of solutions of certain partial differential equations*, Bull. Inst. Math. Acad. Sinica **1** (1973), 93–120.

16. J. B. Serrin, *A uniqueness theorem for a parabolic equation*, Bull. Amer. Math. Soc. **60** (1954), 344 (Abstract).

17. D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. **55** (1944), 85–95. MR **5**, 203.

18. ———, *Positive temperatures on a semi-infinite rod*, Trans. Amer. Math. Soc. **75** (1953), 510–525. MR **15**, 322.

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