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QUASIHARMONIC FUNCTIONS ON THE POINCARÉ N-BALL

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Endowing an abstract Riemann surface with a conformal metric does not affect the harmonicity or the Dirichlet integral of a function on it. A fortiori, the classes O_G, O_{HP}, O_{HB}, O_{HD} of Riemannian 2-manifolds which do not carry Green's functions or harmonic functions which are positive, bounded, or Dirichlet finite, are invariant under varying conformal metrics. Here the harmonicity is defined by $\Delta u = (d\delta + \delta d)u = 0$, with d the exterior derivative, δ the coderivative. In sharp contrast with the harmonic functions, the quasiharmonic functions [8], i.e., solutions of $\Delta u = 1$, on a Riemannian manifold are greatly affected by conformal metrics, and consequently so are the corresponding null classes O_{QP}, O_{QB}, and O_{QD}. A deep and interesting problem is to determine this dependence.

We study the problem in a concrete setup which (a) permits explicit results and (b) yields applications to the general biharmonic classification theory of Riemannian manifolds. The present work is devoted to the former aspect. The latter aspect, an elaborate question in its own right, will be discussed in later studies (e.g. [1], [2], [10]–[16]). Among the phenomena that will be encountered we mention here the following striking result (Hada-Sario-Wang [1]): On the N-ball $|x| < 1$ with the metric $ds = (1 - r^2)\,|dx|$ there exist Dirichlet finite nonharmonic biharmonic functions if and only if $N \leqq 10$.

The Riemannian manifold we choose for our present study of quasiharmonic functions is the N-ball B_α^N with the generalized Poincaré metric

$$ds = (1 - r^2)^\alpha\,|dx|,$$

α a constant. We obtain the following complete characterizations:

$$B_\alpha^N \in O_G \Leftrightarrow \alpha \geqq \frac{1}{N-2},$$

$$B_\alpha^N \in O_{QP} \Leftrightarrow \alpha \notin \left(-1, \frac{1}{N-2}\right),$$

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$$B_\alpha^N \in O_{QP} \Leftrightarrow \alpha \notin \left(-1, \frac{1}{N-2}\right),$$

$$B_\alpha^N \in O_{QD} \Leftrightarrow \alpha \notin \left(-\frac{3}{N+2}, \frac{1}{N-2}\right).$$

The proofs will appear in [**9**].

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