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RESEARCH ANNOUNCEMENTS

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INFINITE SUMS OF PSI FUNCTIONS[1]

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A transformation. The reversible transformation, where $\bar{\lambda} \equiv \lambda + \frac{1}{2}$,

$$\pi C_{2\lambda+1} = 2 \sum_{\kappa=0}^{\infty} \frac{\bar{\lambda} C_{2\kappa}}{\bar{\lambda}^2 - \kappa^2} \quad (\text{all } \lambda), \tag{1a}$$

$$\pi C_{2\kappa} = 2 \sum_{\lambda=0}^{\infty} \frac{\bar{\lambda} C_{2\lambda+1}}{\bar{\lambda}^2 - \kappa^2} \quad (\text{times } \tfrac{1}{2} \text{ if } \kappa = 0) \tag{1b}$$

has the properties[**1**]

$$\sum_{\kappa=0}^{\infty} C_{2\kappa} = 0 \tag{2}$$

if the set $C_{2\lambda+1}$ converges at least like λ^{-t}, $t \geqq 2$, and

$$S = \sum_{\lambda=0}^{\infty} (2\lambda + 1) C_{2\lambda+1} = 0 \tag{3}$$

if the set $C_{2\kappa}$ converges at least like κ^{-r}, $r > 2$.

Consider in particular the elementary sets

$$C_0 = \zeta(r), \quad C_{2\kappa \neq 0} = -\kappa^{-r} \qquad (r = 2, 3, 4, \dots) \tag{4a}$$

which obey (2), and

$$C_{2\lambda+1} = \bar{\lambda}^{-t} \qquad (t = 2, 3, 4, \dots). \tag{4b}$$

For $r = 2$, the sum S is $S_2 = \pi$.

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Applying the transformations (1a, b) to the elementary sets (4a, b), one has

$$\pi C^r_{2\lambda+1} = + \sum_{t=3,5,7,\ldots} (A^r_t/\bar{\lambda}^t) \tag{5}$$

with

$$A^r_t = \begin{Bmatrix} -2\zeta(r+1-t) \\ -4L^*_\lambda \\ +1 \\ 0 \end{Bmatrix} \text{ if } \begin{cases} t < r & \\ t = r & (r \text{ odd only}) \\ t = r+1 & (r \text{ even only}) \\ t > r+1 & \end{cases} \tag{5a}$$

and, using the abbreviation $\bar{\zeta}(n) = \sum_{\lambda=0}^{\infty} (1/\bar{\lambda}^n) = (2^n - 1)\,\zeta(n)$, one has

$$\pi C^t_0 = \bar{\zeta}(t+1), \qquad \pi C^t_{2\kappa \neq 0} = - \sum_{r=2,4,6,\ldots} (B^t_r/\kappa^r) \tag{6}$$

with

$$B^t_r = \begin{Bmatrix} 2\bar{\zeta}(t+1-r) \\ 4L_\kappa \\ 0 \end{Bmatrix} \text{ if } \begin{cases} r < t & \\ r = t & (t \text{ even only}) \\ r > t. & \end{cases} \tag{6a}$$

Here L and L^* represent the ψ-function and may hence be denoted as logarithmic sets:

$$\begin{aligned} L_m &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m-1} = \tfrac{1}{2}[\psi(m+\tfrac{1}{2}) - \psi(\tfrac{1}{2})] \\ &\sim \tfrac{1}{2}(\log m + \gamma) + \log 2 + 1/48m^2 + \cdots \end{aligned} \tag{7}$$

$$L^*_m = L_m + 1/2(2m+1) - \log 2. \tag{7a}$$

Infinite sums. As one applies the summation (2) to the sets $C^t_{2\kappa}$, one regains the known values $\zeta(2n)$ (see [**2**, 23.2.16]), if t is odd, that is, when $C^t_{2\kappa}$ does not contain a logarithmic set. On the other hand, an intriguing sequence of new formulas arises when t is even. For $t = 2$,

$$4 \sum_{k=1}^{\infty} L_k/k^2 = 7\zeta(3). \tag{8}$$

The general formula is

$$4 \sum_{k=1}^{\infty} L_k/k^{2n} = \bar{\zeta}(2n+1) - 2 \sum_{\nu=1}^{n-1} \zeta(2\nu)\bar{\zeta}(2n+1-2\nu). \tag{8a}$$

In a sense, this formula can be considered a formula for $\zeta(2n+1)$ which corresponds to the known formula [2, 23.2.16] for $\zeta(2n)$.

A companion sequence of formulas arises from forming the sum S for the sets $C^r_{2\lambda+1}$. In this case one recovers the known values $\zeta(2n)$ if r is even. For $r = 3$,

$$16 \sum_{k=1}^{\infty} L_k/(2k-1)^2 = 7\zeta(3) + 12\zeta(2)\log 2 \tag{9}$$

and generally

$$4^{n+1} \sum_{k=1}^{\infty} \frac{L_k}{(2k-1)^{2n}} = \bar{\zeta}(2n+1) + 4\bar{\zeta}(2n)\log 2 - 2\sum_{v=1}^{n-1} \bar{\zeta}(2v)\zeta(2n+1-2v). \tag{9a}$$

The two sequences of formulas have considerable similarity. Both are homogeneous in the sum of the arguments in each term if log 2 is written as $\eta(1)$ (see [2, 23.2.19]).

NOTE. The subject reversible transformation arises in the linear theory of a parabolic wing tip in lifting subsonic flow. The fact that it may produce logarithmic sets can be generalized, as follows: If the originating set contains $\log^n$, the logarithmic set in the transformed set is $\log^{n+1}$ if r is odd and if t is even, and is $\log^{n-1}$ if r is even and if t is odd. Detailed derivations are given in [1].

REFERENCES

1. P. F. Jordan, *A reversible transformation and related sets of Legendre coefficients*, AFOSR-TR-72-1706 (1972); RIAS TR 72-14c.

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