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WEAKLY CONTINUOUS ACCRETIVE OPERATORS

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We shall be concerned with the autonomous differential equation

$$u'(t) + Au(t) = 0, \qquad u(0) = x, \tag{1.1}$$

where A is a weakly continuous possibly nonlinear operator mapping a reflexive Banach space X to itself. Recently S. Chow and J. D. Schuur [**2**] have considered existence theory for ordinary differential equations involving weakly continuous operators on separable, reflexive Banach spaces.

We now make clear our notion of strong solutions to (1.1).

DEFINITION 1.2. A function $u:[0, T) \to X$ is said to be a *strong solution* to the Cauchy problem

$$u'(t) + Au(t) = 0, \qquad u(0) = x,$$

provided that u is Lipschitz continuous on each compact subset of $[0, T)$, $u(0) = x$, u is strongly differentiable almost everywhere and $u'(t) + Au(t) = 0$ for a.e. $t \in [0, T)$.

By employing a variant of the Peano method we provide local solution to (1.1).

LEMMA 1.3. *Let X be a reflexive Banach space and suppose that A is a weakly continuous operator with $D(A) = X$. Then there is a finite interval $[0, T)$ such that the Cauchy problem* (1.1) *has a strong solution on* $[0, T)$.

DEFINITION 1.4. An operator A is said to be *accretive* provided that $\|x + \lambda Ax - (y + \lambda Ay)\| \geqq \|x - y\|$ for all $\lambda \geqq 0$ and $x, y \in D(A)$. T. Kato [**5**] has shown that this definition is equivalent to the statement that $\mathrm{Re}(Ax - Ay, f) \geqq 0$ for some $f \in F(x - y)$ where F is the duality map from X to X^*.

If we require that the operator A be accretive we are able to extend the local solution of Lemma 1.3 to a global solution.

THEOREM 1.5. *Let X be a reflexive Banach space and suppose that A is a weakly continuous accretive operator with $D(A) = X$. Then the Cauchy problem* (1.1) *has a unique strong global solution on* $[0, \infty)$.

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If we set $u(t) = T(t)x$ we obtain a semigroup of nonlinear nonexpansive operators $\{T(t): t \geqq 0\}$ which map X to X. We can say that $\{T(t): t \geqq 0\}$ is the semigroup associated with A. The next theorem provides an exponential representation for $\{T(t): t \geqq 0\}$.

THEOREM 1.6. *Let A and X satisfy the conditions of Theorem* 1.5. *Then the operator A is m-accretive, i.e., $R(I + \lambda A) = X$ for all $\lambda \geqq 0$. If $\{T(t): t \geqq 0\}$ is a semigroup associated with A then $T(t)$ may be represented as the pointwise limit*

$$T(t)x = \lim_{n \to \infty} (I + t/nA)^{-n}x.$$

Moreover, for each fixed $t_0 > 0$, the operator $T(t_0)$ is weakly continuous.

The m-accretiveness of A is obtained by considering the equation $u'(t) + A'u(t) = 0$ where $A' = A + I$. Once the m-accretiveness of A has been established the exponential representation of $\{T(t): t \geqq 0\}$ follows immediately from a theorem of M. Crandall and T. Liggett [**1**]. The fact that $T(t_0)$ is weakly continuous is obtained by showing that $(I + \lambda A)^{-1}$ is weakly continuous for all $\lambda \geqq 0$ and employing estimates of Crandall and Liggett. The foregoing results may be applied to the rest point theory developed by C. Yen [**10**].

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