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# FOURIER SERIES OF OPERATORS AND AN EXTENSION OF THE F. AND M. RIESZ THEOREM[1]

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ABSTRACT. A harmonic analysis is defined for operators on $C(\boldsymbol{T})$, the space of all continuous functions on the circle group. An extension of the F. and M. Riesz theorem is obtained.

In this paper we announce several results which will be among those presented in detail in [**2**].

Let $C(\boldsymbol{T})$ be the Banach space of continuous complex valued functions on the circle group $\boldsymbol{T}$ and $\mathscr{L}$ the Banach algebra of all bounded linear operators on $C(\boldsymbol{T})$. For each $t \in \boldsymbol{T}$, the translation operator $R_t$ is defined by

$$(R_t f)(s) = f(s - t).$$

We obtain a representation $\{\Phi_t : t \in \boldsymbol{T}\}$ of the group $\boldsymbol{T}$ on the Banach space $\mathscr{L}$ by defining

$$\Phi_t(T) = R_{-t} T R_t.$$

PROPOSITION 1. *Let $T \in \mathscr{L}$. Then the following are equivalent*:
(a) $\lim_{t \to 0} \|TR_t - R_t T\| = 0$;
(b) $t \rightsquigarrow R_{-t} T R_t$ *is continuous into the norm topology of* $\mathscr{L}$;
(c) $\{R_{-t} T R_t : t \in \boldsymbol{T}\}$ *is separable in the norm topology of* $\mathscr{L}$.

PROOF. This is a consequence of standard group representation arguments.

The operators $T$ in $\mathscr{L}$ which satisfy the conditions of Proposition 1 will be said to *translate continuously*. The collection of operators in $\mathscr{L}$ which translate continuously will be denoted by $\mathscr{L}_{\#}$. $\mathscr{L}_{\#}$ is a norm closed, $\{\Phi_t\}$-invariant, subalgebra of $\mathscr{L}$.

Let $M(\boldsymbol{T})$ be the space of all finite Borel measures on $\boldsymbol{T}$. For each $\mu$ in $M(\boldsymbol{T})$, the convolution operator $C_\mu$ on $C(\boldsymbol{T})$ is defined by

$$(C_\mu f)(s) = \int_{\boldsymbol{T}} f(s - t)\, d\mu(t).$$

For each integer $n$, the operator $M_n$ on $C(\boldsymbol{T})$ is defined by

$$(M_n f)(s) = e^{ins} f(s).$$

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For each integer $n$, we denote by $\mathscr{L}_n$ the linear subspace of $\mathscr{L}$ consisting of those $T$ which satisfy the commutation relations

$$TR_t = e^{int}R_tT, \qquad t \in \boldsymbol{T}.$$

It is clear that each $\mathscr{L}_n$ is contained in $\mathscr{L}_\#$, $\mathscr{L}_m \cap \mathscr{L}_n = \{0\}$ if $m \neq n$ and $ST \in \mathscr{L}_{m+n}$ if $S \in \mathscr{L}_m$ and $T \in \mathscr{L}_n$.

It is known that $\mathscr{L}_0$, the set of operators in $\mathscr{L}$ which commute with translations, is identical with $\{C_\mu : \mu \in M(\boldsymbol{T})\}$ (see [**3**]).

**Proposition** 2. *Let $T \in \mathscr{L}$. Then the following are equivalent*:

(a) $T \in \mathscr{L}_n$;

(b) *there is $\mu$ in $M(\boldsymbol{T})$ so $T = M_n \cdot C_\mu$*;

(c) *there is $\lambda$ in $M(\boldsymbol{T})$ so $T = C_\lambda \cdot M_n$.*

**Proof.** This is an easy consequence of $\mathscr{L}_0 = \{C_\mu : \mu \in M(\boldsymbol{T})\}$ and the fact that $M_n \in \mathscr{L}_n$.

For each integer $n$, define the operator $\pi_n : \mathscr{L}_\# \to \mathscr{L}_\#$ by the $\mathscr{L}_\#$-valued integral

$$\pi_n(T) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-int} R_{-t} T R_t \, dt.$$

It is easy to check that $\pi_n$ is a projection of $\mathscr{L}_\#$ onto $\mathscr{L}_n$.

For each $T$ in $\mathscr{L}_\#$, the formal series $\sum_{-\infty}^{+\infty} \pi_n(T)$ will be called the *Fourier series* of $T$. (If $T$ is the operation of multiplication by a continuous function $\phi$, its Fourier series is $\sum_{-\infty}^{+\infty} \hat{\phi}(n) M_n$.)

**Proposition** 3. *Let $T$ be a bounded linear operator on $C(\boldsymbol{T})$ that translates continuously. Then the Fourier series of $T$ is $C - 1$ summable to $T$ in the norm topology of $\mathscr{L}$.*

**Proof.** This is immediate from Theorem 1.1 of [**1**].

**Corollary** 4. *$\mathscr{L}_\#$ is the smallest norm closed subalgebra of $\mathscr{L}$ containing $\{M_n : n \in \boldsymbol{Z}\}$ and $\{C_\mu : \mu \in M(\boldsymbol{T})\}$.*

As an application of Proposition 3 we obtain a generalization of the F. and M. Riesz theorem. Define $C(\boldsymbol{T})_+$ and $C(\boldsymbol{T})_-$ by

$$C(\boldsymbol{T})_+ = \{f : f \in C(\boldsymbol{T}), \hat{f}(n) = 0 \text{ if } n < 0\},$$
$$C(\boldsymbol{T})_- = \{f : f \in C(\boldsymbol{T}), \hat{f}(n) = 0 \text{ if } n > 0\}.$$

**Proposition** 5. *Let $T$ be a bounded linear operator on $C(\boldsymbol{T})$ that translates continuously. If $T(C(\boldsymbol{T})_+) \subseteq C(\boldsymbol{T})_-$, then $T$ must be a compact operator.*

In the case that $T = C_\mu$, for some measure $\mu$, because of Lemma 6 below, Proposition 5 reduces to the F. and M. Riesz theorem, for $C_\mu(C(\boldsymbol{T})_+) \subseteq C(\boldsymbol{T})_-$ is equivalent to $\hat{\mu}(n) = 0$ for $n > 0$.

LEMMA 6. *Let $\mu \in M(\boldsymbol{T})$. Then the following are equivalent:*
(a) *$\mu$ is absolutely continuous with respect to Lebesgue measure;*
(b) *the convolution operator $C_\mu$ is compact.*

PROOF. This is an easy consequence of the Ascoli theorem and the fact that a measure $\mu$ is absolutely continuous if and only if $\lim_{t\to 0}\|R_t^*\mu - \mu\| = 0$.

PROOF OF PROPOSITION 5. Let $n$ be any positive integer. $\pi_n(T)C(\boldsymbol{T})_+ \subseteq C(\boldsymbol{T})_-$ follows by computation from $T(C(\boldsymbol{T})_+) \subseteq C(\boldsymbol{T})_-$. By Proposition 2, $\pi_n(T) = C_\lambda \circ M_n$ for some $\lambda \in M(\boldsymbol{T})$. Thus $C_\lambda(M_n(C(\boldsymbol{T})_+)) \subseteq C(\boldsymbol{T})_-$, which shows that $\lambda$ must be absolutely continuous, because of the F. and M. Riesz theorem. By Lemma 6, $\pi_n(T) = C_\lambda \circ M_n$ is compact, and Proposition 5 now follows from Proposition 3.

Analogues of the above results are valid in rather general circumstances. $C(\boldsymbol{T})$ can be replaced by any homogeneous Banach space of functions (in the sense of [**3**]) which is closed under multiplication by exponentials, and analogues of Propositions 1, 2, and 3 will hold. Proposition 5 remains valid if $C(\boldsymbol{T})$ is replaced by $L^1(\boldsymbol{T})$. The formal Fourier series we have defined makes sense for any operator $T$ on a homogeneous Banach space, even if $T$ does not translate continuously. The integral defining $\pi_n(T)$ will exist in the strong operator topology and the Fourier series of $T$ is summable to $T$ in that topology. Finally, analogues for compact abelian groups can be established. These matters, together with formal properties of the Fourier series, will be discussed in [**2**].

## BIBLIOGRAPHY

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