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AN EXPOSITION OF THE STRUCTURE OF SOLVMANIFOLDS. PART II: G-INDUCED FLOWS[1]

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CONTENTS

1. **Introduction.** During the academic year 1960–1961 there was a conference held in topological dynamics at Yale University at which the study of G-induced flows on solvmanifolds was initiated and which resulted in the publication of [**1**]. At the conference held on topological dynamics at Yale in June 1972 to honor Professor G. H. Hedlund on his retirement, it was my pleasure to finally announce [**5**] the first theorem giving a necessary and sufficient condition for the ergodicity of a G-induced flow on a compact solvmanifold. Since this result draws on material that are parts of papers written during the intervening twelve years it seems appropriate to take the time and effort to present a fairly systematic and self-contained account of this result. This is particularly true in view of the fact that we are now in possession of an algebraic machine that gives vastly simpler proofs of the required algebraic results. This algebraic material has itself only recently been given a unified exposition in [**4**].

There is always, in writing a paper of this sort, the problem of what to assume and what to present. Here, since we are trying to cover so much material if we tried to be complete we would require a paper of considerable length. However, we also have the perspective of time that tells us that it is the solvable theory that is not understood and that the nilpotent theory is readable even in its original version. Hence we have contented

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ourselves with giving a brief summary of the nilpotent theory and concentrated our efforts on the solvable theory. It seems that [**3**] is particularly impenetrable and so we have reproduced new proofs for the results from [**3**].

Let us now begin by establishing a language in which to state the three main problems that we wish to solve in this paper. Let R be a connected, simply connected solvable Lie group and let D be a closed subgroup. The homogeneous space R/D is called a solvmanifold. If R/D is compact it is well known that R/D has a unique probability measure invariant under R. Whenever we talk about a flow on R/D being ergodic it will be with respect to this measure. Let $R/D = X$ be a compact solvmanifold. Then R/D is called a presentation of X if D contains no connected subgroup normal in R. Let $p(\xi)$, $\xi \in \boldsymbol{R}$, be a one parameter subgroup of R and let $p(\xi)$ act on the coset γD by sending it to the coset $(p(\xi)\gamma)D$. We will call this a G-induced flow (an abbreviation for group generated flow) on R/D.

Now it can happen that R_1/D_1 and R/D can be presentations of the same compact solvmanifold without R_1 being isomorphic to R. Indeed if X is a compact solvmanifold it has an infinite number of presentations R_α/D_α with $\lim_{\alpha\to\infty} \dim R_\alpha = \infty$.

Before stating our three main problems let us consider some examples that show that they are not trivial.

EXAMPLE. Let $\mathscr{R}$ be the universal covering group of the group of rigid motions of the euclidean plane. Then $\mathscr{R}$ has a matrix representation of the form

$$\begin{pmatrix} \cos 2\pi t & \sin 2\pi t & 0 & r \\ -\sin 2\pi t & \cos 2\pi t & 0 & s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r, s, t \in \boldsymbol{R}.$$

Let us view (r, s, t) as the coordinates of a point in $\mathscr{R}$ and let Γ be the subgroup of elements of the form $(n_1 + n_3 u, n_2 + n_3 v, n_3)$, $n_i \in \boldsymbol{Z}$, $i = 1, 2, 3$. Then $\mathscr{R}/\Gamma$ is topologically the three-dimensional torus and Γ is isomorphic to the integers taken three times.

The Lie algebra of $\mathscr{R}$, $L(\mathscr{R})$, consists of matrices of the form

$$\begin{pmatrix} 0 & c & 0 & a \\ -c & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a, b, c \in \boldsymbol{R}.$$

Let $p(\xi)$, $\xi \in \boldsymbol{R}$, be a one parameter subgroup of $\mathscr{R}$ and let X be the tangent vector to $p(\xi)$ at the identity. Then $X \in L(\mathscr{R})$ and let $X = (a, b, c)$ with $c \neq 0$. Then we may choose $x = (r_0, s_0, 0) \in \mathscr{R}$ such that $\operatorname{ad}(x)X = (0, 0, c)$. But $\operatorname{ad}(x)\Gamma = \Gamma$. Hence either $p(\xi)$ lies in the subgroup $A = \{(r, s, 0) \in \mathscr{R} | r, s \in \boldsymbol{R}\}$ and is easily seen not to be ergodic or the flow induced by $p(\xi)$ on $\mathscr{R}/\Gamma$ is the same as the flow induced by the subgroup $p_0(\xi) = \{(0, 0, \xi) \in \mathscr{R} | \xi \in \boldsymbol{R}\}$.

Now consider the mapping

$$l : \mathscr{R} \to \mathscr{R}$$

which assigns to each element of $\mathscr{R}$ its inverse. This maps the coset $x\Gamma \to \Gamma x^{-1}$ and is a homeomorphism of $\mathscr{R}/\Gamma$ with $\Gamma\backslash\mathscr{R}$ and if we map $p_0(\xi)$ to $p_0(-\xi)$ we have an equivalent flow. Now consider the mapping

$$\phi : \mathscr{R} \to R^3$$

that assigns to $(r, s, t) \in \mathscr{R}$ the points in R^3 with these coordinates. Since $(r, s, t)(0, 0, t_0) = (r, s, t + t_0)$ we easily see that $p_0(\xi)$ acting on $\Gamma\backslash\mathscr{R}$ is equivalent to linear action on R^3. Now ϕ induces a homeomorphism ϕ^* of $\Gamma\backslash\mathscr{R}$ with $\Gamma\backslash R^3$, the three-dimensional torus.

It is now classical that the flow $p_0(\xi)$ is ergodic if and only if $u, v, 1$ are rationally independent. Hence if $\Gamma_0 = (n_1, n_2, n_3)$, no G-induced flow is ergodic.

Thus the three-dimensional torus has presentations with and without G-induced ergodic flows.

Now consider the subgroup $\Gamma_{1/2}$ of $\mathscr{R}$ generated by the three elements $(1, 0, 0)$, $(0, 1, 0)$, $(\alpha, \beta, \frac{1}{2})$. By conjugating by elements of A we may replace the generators of $\Gamma_{1/2}$ by $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, \frac{1}{2})$. But then $\Gamma_{1/2}$ is finitely covered by $\mathscr{R}/\Gamma_0$ where

$$\Gamma_0 = (n_1, n_2, n_3), \qquad n_i \in Z.$$

Since $\mathscr{R}/\Gamma_0$ has no G-induced ergodic flow—indeed the flow is periodic—it follows easily that $\mathscr{R}/\Gamma_{1/2}$ has no G-induced ergodic flow. We will ultimately see that *no presentation* of $\mathscr{R}/\Gamma_{1/2}$ has a G-induced ergodic flow.

With these examples as motivation let us now state our three main problems.

Problem 1. Given $p(\xi)$ a G-induced flow on a presentation R/D of a compact solvmanifold, give a necessary and sufficient condition for $p(\xi)$ to act ergodically on R/D.

The solution of Problem 1 is given in Theorem A of §6.

Our proof of Theorem A is fairly self-contained and can be read by anyone with a knowledge of Chapters I and II and §2 of Chapter III of [**4**] and facts about flows on nilmanifolds.

Problem 2. Given a compact solvmanifold X, give a necessary and sufficient condition for X to have a presentation R/D such that R/D has an ergodic G-induced flow.

The solution of Problem 2 is given in Theorem B of §7.

Unfortunately, the solution of this problem and that of Problem 3 below requires fairly detailed knowledge from [**4**].

Problem 3. Given a compact solvmanifold X, give a necessary and sufficient condition for X to have a presentation R/D such that R/D has a minimal G-induced flow.

(Recall that a flow is called minimal if the closure of *every* orbit is X.)

2. **Abelian and nilpotent theory.** Although by now the abelian and nilpotent theories of G-induced flows are well understood, let me take this opportunity to give a new formulation of the results that stresses the role of rational algebraic group theory.

Let V^n be an n-dimensional real vector space and let $e_1, \dots, e_n$ be a basis of V^n. Let $V^n_{\mathbf{Q}} = \{\mathbf{v} \in V^n | \sum q_i e_i, q_i \in \mathbf{Q}\}$. We will call $V^n_{\mathbf{Q}}$ a rational form of V^n. Clearly $V^n_{\mathbf{Q}}$ is an n-dimensional rational vector space and the integer linear combinations L of any basis of $V^n_{\mathbf{Q}}$ will be called a lattice in V^n. In other words L is a discrete subgroup of V^n such that V^n/L is compact. Let A be a linear subspace (real) of V^n. Let LA denote the subgroup of V^n generated by the elements of L and A and let $((LA)^-)_0$ denote the identity component of the closure of the group LA. We wish to characterize the group $((LA)^-)_0$. To do this, we will introduce the following definition.

DEFINITION 2.1. Let X be a subset of V^n. The *rational algebraic group hull* of X, denoted by $A^{\mathbf{Q}}_h(X)$, is defined by the formula

$$A^{\mathbf{Q}}_h(X) = \bigcap W$$

where W is a real vector space which contains X and is such that $W \cap V^n_{\mathbf{Q}}$ is a rational form of W.

It is an elementary exercise to verify that $((LA)^-)_0 = A^{\mathbf{Q}}_h(A)$ and further that if $p(\xi)$ is a one parameter subgroup of V^n then $p(\xi)$ acts ergodically on V^n/L, where L is a lattice in $V^n_{\mathbf{Q}}$, if and only if $A^{\mathbf{Q}}_h(p(\xi)) = V^n$. *If $p(\xi)$ has the property that $A^{\mathbf{Q}}_h(p(\xi)) = V^n$, we will say that $p(\xi)$ is in general position relative to the rational form $V^n_{\mathbf{Q}}$.*

Now let N be a nilpotent Lie group (connected, and simply connected will always be assumed). Then we may find a global coordinate system $(x_1, \dots, x_n)$ for N such that if $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are elements of N then $\mathbf{x} \cdot \mathbf{y} = (p_1(\mathbf{x}, \mathbf{y}), \dots, p_n(\mathbf{x}, \mathbf{y}))$, where the $p_i(\mathbf{x}, \mathbf{y})$ and polynomials in $2n$ variables and the dot denotes group multiplication in N. Now N has a discrete cocompact subgroup Γ if and only if we may choose a coordinate system $(x_1, \dots, x_n)$ for N such that

1. all the polynomials defining group multiplication in N with coordinate system $(x_1, \dots, x_n)$ have rational coefficients;

2. if N_Q is the subgroup of N consisting of all the points of N with rational coordinates in the above coordinate system then $\Gamma \subset N_Q$.

Whenever $N_Q \subset N$ satisfies 2 above we will call it a rational form of N.

We may now generalize Definition 2.1 to the nilpotent case.

DEFINITION 2.2. Let N be a connected, simply connected nilpotent group with rational form N_Q and let X be a subset of N. The rational algebraic group hull of X, denoted by $A_h^Q(X)$, is defined by the formula

$$A_h^Q(X) = \bigcap W$$

where W is a connected subgroup of N which contains X and is such that $W \cap N_Q$ is a rational form of W.

Let $p(\xi)$ be a one parameter subgroup of N. We will say that $p(\xi)$ is in general position relative to the rational form N_Q if and only if $A_h^Q(p(\xi)) = N$.

Let us begin by proving the following elementary property of rational algebraic group hulls that we will need in our later discussion.

THEOREM 2.1. *Let G be a connected normal subgroup of N and let N_Q be a rational form of N. Then $A_h^Q(G)$ is a connected normal subgroup of N.*

PROOF. Let $x \in N_Q$. Then $\mathrm{ad}(x)$ maps $L(G) \subset L(N)$ onto itself. Hence if $L(W)$ is the Lie algebra of any subgroup W such that $W \cap N_Q$ is a rational form of W and $L(W) \supset L(G)$, then $\mathrm{ad}(x)(L(W)) \supset L(G)$ and has the property that $\mathrm{ad}(x)(W) \cap N_Q$ is a rational form of $\mathrm{ad}(x)(W)$. Hence

$$\mathrm{ad}(x)(A_h^Q(G)) = A_h^Q(G), \qquad x \in N_Q.$$

Since N_Q is dense in N and the adjoint mapping is continuous we have that $A_h^Q(G)$ is normal in N.

Let us now relate this to flows on nilmanifolds.

It is an elementary argument that shows that if N has a rational form N_Q then $[N, N] \cap N_Q$ is a rational form for $[N, N]$, where $[N, N]$ is the commutator subgroup of N. From this it follows easily that N_Q determines a unique rational form on the vector space $N/[N, N]$ which we will call the induced rational form.

THEOREM 2.2. *Let N be a nilpotent Lie group with rational form N_Q and let $p(\xi)$ be a one parameter subgroup of N. Then $p(\xi)$ is in general position in N if and only if the image of $p(\xi)$ is in general position in $N/[N, N]$ relative to the rational form induced by N_Q.*

PROOF. Assume $p(\xi)$ is not in general position in N. Then there exists a nontrivial $M = A_h^Q(p(\xi))$. Let

$$\alpha: N \to N/[N, N]$$

be the canonical mapping. Since $[N, N] \cap N_Q$ is a rational form of $[N, N]$ it is easy to verify that $\alpha(M) = A_h^Q(\alpha(p(\xi)))$. Since $M \neq N$ it is well known that $\alpha(M) \neq N/[N, N]$ and so $\alpha(p(\xi))$ is not in general position in $N/[N, N]$.

Assume now that $\alpha(p(\xi))$ is not in general position in $N/[N, N]$. Then $\alpha^{-1}(A_{\mathfrak{n}}^{Q}(\alpha(p(\xi)))) \subset A_{\mathfrak{n}}^{Q}(p(\xi))$ and does not equal N. Hence $p(\xi)$ is not in general position in N. This proves our assertion.

From Chapter V of [**1**] we may now conclude the following:

THEOREM 2.3. *Let N/Γ be a compact nilmanifold with Γ a discrete subgroup of the nilpotent Lie group N. Then $p(\xi)$ acts ergodically on N/Γ if and only if $p(\xi)$ is in general position relative to the rational form N_Q containing Γ.*

From Chapter IV of [**1**] we have

THEOREM 2.4. *Let N/Γ be a compact nilmanifold and let $p(\xi)$ be a one parameter subgroup of N. Then $p(\xi)$ acts distally on N/Γ and acts minimally on N/Γ if and only if it acts ergodically on N/Γ.*

THEOREM 2.5. *Let N be a nilpotent Lie group and let Γ_1 and Γ_2 be discrete subgroups of N such that N/Γ_1 and N/Γ_2 are compact. Let $p(\xi)$ be a one parameter subgroup of N. If $\Gamma_1/\Gamma_1 \cap \Gamma_2$ and $\Gamma_2/\Gamma_1 \cap \Gamma_2$ are finite then $p(\xi)$ acts ergodically (minimally) on N/Γ_1 if and only if its acts ergodically (minimally) on N/Γ_2.*

PROOF. Since $\Gamma_1/\Gamma_1 \cap \Gamma_2$ is finite, if N_Q is the rational form of N containing Γ_1 then $N_Q \supset \Gamma_2$. Hence, by Theorem 2.3, $p(\xi)$ acts ergodically on N/Γ_1 if and only if it acts ergodically on N/Γ_2. Theorem 2.4 then implies the result on minimality.

3. **Algebraic theory of compact solvmanifolds.** In [**4**] we presented a general study of the algebraic theory of solvmanifolds. This treatment, because it includes a study of noncompact solvmanifolds, existence and uniqueness theorems for compact solvmanifolds and applications, does not give the most succinct account of the algebraic results we will need to settle Problem 1. Accordingly, we will, assuming that the reader is familiar with Chapters I and II and §2 of Chapter III of [**4**], present a self-contained account of most of the algebraic results we will need up to §6 of this paper. However, the proofs, although not the statement of results in §7 will need the full strength of results from [**4**] including existence theorems and discrete semisimple splittings.

We will henceforth adopt the notion $A \rtimes B$ or $B \ltimes A$ to denote the semidirect product of A and B, where A is the normal subgroup. If $\mathscr{A}(A)$ denotes the group of automorphisms of A, and $\alpha: B \to \mathscr{A}(A)$ is a homomorphism, we will sometimes use $A \rtimes (B)\alpha$ or $(B)\alpha \ltimes A$ to denote the semidirect product of A and B, where B acts on A through $\alpha(B)$.

Now let R be a connected, simply connected solvable Lie group with semisimple splitting

$$R_S = R \rtimes T_R = M_R \rtimes T_R$$

where M_R is the nil-radical of R_S and T_R is an abelian semisimple group of automorphisms of M_R. One of the important properties of R_S, implicitly used in [**4**], is given in the following theorem whose proof will review the construction of R_S.

THEOREM 3.1. *Let G be a connected, normal subgroup of R and let R_S be the semisimple splitting of R. Then G is a normal subgroup of R_S.*

PROOF. Let $L(R)$ denote the Lie algebra of R and let $\mathrm{ad}(R)$ denote the adjoint representation of R. Then $L(G) \subset L(R)$ is invariant under $\mathrm{ad}(R)$ and hence under $A_h(\mathrm{ad}(R))$ where $A_h(\quad)$ denotes the algebraic hull of the group in the bracket. By definition $T_R \subset A_h(\mathrm{ad}(R))$ and so $L(R)$ is invariant under T_R. Since $L(G)$ is invariant under $\mathrm{ad}(R)$ also, and $R_S = R \rtimes T_R$, it follows that $L(G)$ is invariant under $\mathrm{ad}(R_S)$ and so G is normal in R_S.

Let $R_S = M_R \rtimes T_R$ be a semidirect product representation of R_S. This representation is of course not unique in that we will get the same group R_S for many different homomorphisms of T_R into the automorphism group of M_R, $\mathscr{A}(M_R)$. We will now see that if $p(\xi)$ is a one parameter subgroup of R then certain representations of R_S as a semidirect product have a nicer structure relative to $p(\xi)$ than other representations.

Let $X \in L(R)$ be the tangent vector to $p(\xi)$ at the origin and consider $\mathrm{Ad}(X)$ acting on $L(R)$. By the Jordan canonical form theorem $\mathrm{Ad}(X) = t' + n$, $t'n = nt'$ and n is nilpotent and t' is semisimple. Clearly

$$\exp(\mathrm{Ad}(\xi X)) \subset \mathrm{ad}(R) \quad \text{and} \quad \exp(\mathrm{Ad}(\xi X)) = t(\xi)u(\xi),$$

$t(\xi)$, $u(\xi)$ one parameter groups, where $t(\xi)u(\eta) = u(\eta)t(\xi)$, $n, \xi \in \mathbf{R}$; $u(\xi)$ is unipotent and $t(\xi)$ is semisimple. But since $\exp(\mathrm{Ad}(\xi X)) \subset \mathrm{ad}(R)$ we have that $u(\xi)$ and $t(\xi)$ are contained in $A_h(\mathrm{ad}(R))$. By the Mostow conjugacy theorem for maximal ad-reductive subgroups we may choose T_R to contain $t(\xi)$. Hence we have

1. $p(\xi) = t(\xi)m(\xi)$, $t(\xi)$, $m(\xi)$ are one parameter groups,
2. $t(\xi)m(\eta) = m(\eta)t(\xi)$, $\eta, \xi \in \mathbf{R}$,
3. $t(\xi) \subset T_R$ and $m(\xi) \subset M_R$.

We will call $t(\xi)$ the semisimple part of $p(\xi)$ and $m(\xi)$ the unipotent part of $p(\xi)$ and it is a classical result that they are indeed unique.

Henceforth, we will always assume that if $p(\xi)$ is a one parameter subgroup of R and $R_S = M_R \rtimes T_R$ is the semisimple splitting of R then the semidirect product representation of R_S has been chosen so that 3 *above is satisfied.*

Now consider a compact solvmanifold R/D. (At this stage we do not need to assume there is a presentation.) Let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R. Then for each $d \in D$ we have $d = m(d)t(d)$, where $m(d) \in M_R$ and $t(d) \in T_R$. We may now define D as a group acting on M_R; we will call this the *affine action* of D on M_R, as follows:

Let $m \in M_R$ and let $d \in D$. We define

$$\alpha(d)(m) = m(d)(t(d)(m))$$

where $t(d)(m)$ denotes the image of m in M_R under $t(d)$ considered as an automorphism of M_R. Now let

$$\pi : R \to M_R$$

be the mapping obtained by restricting the projection mapping $M_R \rtimes T_R \to M_R$ to R. It is well known [4] and easily verified that π is a homeomorphism. Further, if left multiplication by $d \in R$ acting on R is denoted by l_d then the following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "l_d"] & M_R \arrow[d, "\alpha(d)"] \\ R \arrow[r, "\pi"] & M_R \end{tikzcd}$$

From this it follows easily that π induces a homeomorphism

$$\pi^* : R/D \to M_R/\alpha(D)$$

where $M_R/\alpha(D)$ denotes the orbit space of M_R under the affine action of D.

Now let $\mathscr{M}_D = \{m(d) \in M_R | \alpha \in D\}$. Since $\mathscr{M}_D \subset M_R$ we may consider the smallest connected subgroup of M_R that contains $\mathscr{M}_D$ and call this group the algebraic hull of $\mathscr{M}_D$ and denote it by $A_h(\mathscr{M}_D)$. It is known that R/D is compact if and only if $A_h(\mathscr{M}_D) = M_R$. However, we will only need and only prove the easier half of this equivalence.

THEOREM 3.1. *Let R/D be a compact solvmanifold and let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R. For $d \in D$, let $m(d)$ be the M_R component of d and let $\mathscr{M}_D = \{m(d) | d \in D\}$. Then the algebraic hull of $\mathscr{M}_D$ in M_R, $A_h(\mathscr{M}_D)$,* equals M_R.

PROOF. We note that $L(A_h(\mathscr{M}_D))$ is $\mathscr{T}_D$ invariant, where, if $t(d)$ denotes the T_R component of $d \in D$, $\mathscr{T}_D = \{t(d) | d \in D\}$. Since $\mathscr{T}_D$ is isomorphic to a homomorphic image of D, $\mathscr{T}_D$ is a subgroup of T_R. Since $\mathscr{T}_D$ is semisimple there exists a vector space V, of $L(M_R)$ that is $\mathscr{T}_D$ invariant and such that

$$L(M_R) = L(A_h(\mathscr{M}_D)) \oplus V.$$

Let $E(V) \subset M_R$ denote the image of V under the exponential mapping.

Consider the disjoint subsets of M_R given by $\delta E(V)$, $\delta \in A_h(\mathscr{M}_D)$. Observe that $\alpha(d)(\delta E(V)) = \delta' E(V)$, where $\delta' \in A_h(\mathscr{M}_D)$. Hence $\delta E(V)$ determines a fibration of $R/D = M_R/\alpha(D)$ with euclidean space as fiber. Since R/D is compact, this is possible only if $E(V)$ consists of one point and $A_h(\mathscr{M}_D) = M_R$.

Theorem 3.1 has a symmetric version involving T_R as given below:

THEOREM 3.2. *Let R/D be a compact solvmanifold and let $R_S = M_R \rtimes T_R$ be a semisimple splitting of R. For $d \in D$ let $t(d)$ denote the T_R component of d and let $\mathscr{T}_D = \{t(d) | d \in D\}$. Then $T_R/\mathscr{T}_D^-$ is compact, where the bar denotes the closure operation.*

PROOF. Let N be the nil-radical of R. Then R_S/N is abelian and if we use $\sim$ to denote universal covering group

$$R_S/N^\sim = M_R/N \oplus T_R^\sim .$$

Hence $T_R^\sim$ is a vector space of the same dimensions as M_R/N. Further R/N may be viewed as the diagonal in $R_S/N^\sim$ relative to the above direct sum representation. Hence the algebraic hull of the image of D in $T_R^\sim$ must equal $T_R^\sim$ by Theorem 3.1. This proves our assertion.

We will need one result from [**6**] that we will state without proof. The reasons for this are two-fold: First, the proof is technical and I cannot improve on the version in [**6**]; second, the result is intuitively reasonable.

Let R be a connected, simply connected solvable Lie group with nil-radical N and let D be a closed subgroup of N. Then $D \cap N$ is a closed subgroup of N and D satisfies the exact sequence

$$1 \to D \cap N \to D \to \mathbf{Z}^s \oplus \mathbf{R}^t \to 1$$

where s and t are integers greater than or equal to zero.

The only content of this result is that $s \neq \infty$.

Before discussing the Mostow structure theorem for presentations of compact solvmanifolds let us state two lemmas that will be useful later on.

LEMMA 3.3. *Let D be a closed subgroup of a connected, simply connected solvable Lie group R, let D_0 be the identity component of D and let $\mathscr{M}_D$ and $\mathscr{T}_D$ be as defined above. Let*

$$G = A_h(\mathscr{M}_D) \rtimes A_h(\mathscr{T}_D)$$

where $A_h(\mathscr{T}_D)$ is the algebraic hull of $\mathscr{T}_D$ in $\mathscr{A}(M_R)$. Then D_0 is invariant under inner automorphisms of R by elements of G.

PROOF. Consider $L(R)$ and $L(D_0) \subset L(R)$. Since $L(D_0)$ is invariant under $\operatorname{ad}(D)$, it is invariant under $A_h(\operatorname{ad}(D))$. But $A_h(\operatorname{ad}(D)) \supset A_h(\mathscr{M}_D) \rtimes A_h(\mathscr{T}_D)$.

LEMMA 3.4. *Viewing T_R and $\mathscr{T}_D$ in $\mathscr{A}(M_R)$ let A be the subgroup generated by $A_h(\mathscr{T}_D)$ and T_R. Then*

$$A = C \oplus A_h(\mathscr{T}_D)$$

where C is a compact abelian group.

PROOF. Clearly $A_h(\mathscr{T}_D) \supset \mathscr{T}_D^-$ and since $T_R/\mathscr{T}_D^-$ is compact and T_R and $A_h(\mathscr{T}_D)$ generate an abelian group, we have our assertion.

The Mostow structure theorem for presentations of compact solvmanifolds has two equivalent formulations as follows:

Let R/D be a presentation of a compact solvmanifold, let N be the nil-radical of R and let D_0 be the identity component of D. Then $D_0 \subset N$ and DN/N is a discrete cocompact subgroup of R/N.

Let R/D be a presentation of a compact solvmanifold and let $\tau : R \to T_R$ be the homomorphism induced from the homomorphism $R_S \to R_S/M_R \approx T_R$. Then $\tau(D)$ is a discrete cocompact subgroup of T_R.

Until now we have denoted $\tau(D)$ by $\mathscr{T}_D$. But now we will change notation and denote it by T_D. We will now define

$$D_S = M_R \rtimes T_D$$

and call this group the *semisimple splitting of D*. Clearly $D_S \supset D$ and D_S/D is compact. (We are not quite following the language or notation of [**4**] at this point.)

We will now introduce a class of solvable Lie groups that will play an important role in our discussion. Let R be a connected, simple connected solvable Lie group and let $R_S = M_R \rtimes T_R$ be its semisimple splitting. We will say that R is *class R* if and only if all the eigenvalues of elements of T_R have absolute value one. If R/D is the presentation of a compact solvmanifold and if R is class R we will call the solvmanifold R/D a *class R compact solvmanifold.*

Let us now establish some important properties of class R solvmanifolds.

THEOREM 3.5. *Let R/D be the presentation of a compact class R solvmanifold and let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R. Then T_R is closed in $\mathscr{A}(M_R)$ (and hence compact) and $D/D \cap M_R$ is finite. Further, if $\pi : R \to M_R$ comes from the projection mapping $M_R \rtimes T_R \to M_R$, then π induces a homeomorphism $\pi^* : R/D \cap M_R \to M_R/D \cap M_R$.*

PROOF. Define $R_S^* = M_R \rtimes T_R^-$, where the closure of T_R is taken in $\mathscr{A}(M_R)$. Then T_R^-, being a closed subgroup of a compact group, (recall, that all the eigenvalues of elements of T_R have absolute value one) is compact. Let R^* denote the universal covering group of R_S^* and let D^* be the pre-image of D in R^*. Since R_S^*/D is compact, R^*/D^* is compact. Now the nil-radical of R^* is M_R. By the Mostow structure theorem, $M_R/D^* \cap M_R$ is compact. Hence $M_R/D \cap M_R$ is compact.

Recall that if $\pi : R \to M_R$ is the mapping induced by the projection mapping $M_R \rtimes T_R \to M_R$ then π induces a homeomorphism

$$\pi^*: R/D \cap M_R \to M_R/D \cap M_R.$$

Hence $R/D \cap M_R$ is compact and $D/D \cap M_R$ is finite. Notice that $M_R/D \cap M_R$ is a compact nilmanifold.

Let us now look at a theorem that goes in the converse direction.

THEOREM 3.6. *Let R/D be a presentation of a compact solvmanifold with nilpotent fundamental group. Then D is nilpotent and R is class R.*

PROOF. Let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R and let $D_S = M_R \rtimes T_D$ be the semisimple splitting of D. Recall that D_0, the identity component of D, is continued in M_R. Then, as one proved Theorem 3.1, it is easy to verify that D_0 is normal in D_S. Because D/D_0 is nilpotent, being the fundamental group of R/D, we have easily that D_S/D_0 is nilpotent or that

$$D_S/D_0 = M_R/D_0 \oplus T_D.$$

Thus to show that D is nilpotent amounts to showing that T_D acts trivially on D_0.

To show that T_D acts trivially on D_0, let $V \subset D_0$ be a primary subspace of D_0 relative to $t \in T_D$, $t \neq$ identity. Assume that the eigenvalue of t restricted to V is not one. Then V is a primary subspace of t acting on M_R, because, on M_R/D_0, t acts trivially. Then since T_D and T_R commute, V is invariant under the action of T_R. Now let $I(V)$ be the normal subgroup generated by V in M_R. Then $I(V) \subset D_0$ and is T_R invariant. Hence $I(V)$ is normal in R_S and so normal in R since it is contained in R. This contradicts the statement that R/D is a presentation and so proves the first assertion of our theorem and shows also that $D_S = M_R$ and $D \subset M_R$.

It remains to show that R is class R or equivalently that T_R is compact. By Theorem 3.2, T_R/T_D^- is compact. But T_D^- is the identity and so T_R is compact.

4. **An equivalence theorem for G-induced flows.** In §3 we established the fundamental fact that every class R solvmanifold is finitely covered by a compact nilmanifold. Let us now see that G-induced flows on class R solvmanifolds also are related to G-induced flows on nilmanifolds.

THEOREM 4.1. *Let $p(\xi)$ be a G-induced flow in general position in the class R compact solvmanifold R/D. Then the flow induced by $p(\xi)$ on $R/D \cap M_R$ is equivalent to the unipotent part of $p(\xi)$, $m(\xi)$, acting on $M_R/D \cap M_R$.*

PROOF. Let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R. Then we know that T_R is a compact group and by our convention the representation of the semidirect product has been chosen so that $m(\xi) \subset M_R$, $t(\xi) \subset T_R$, each is a one parameter group, the group they generate is abelian and

$$p(\xi) = m(\xi)t(\xi), \qquad t \in \boldsymbol{R}.$$

Since $p(\xi)$ is in general position in R, $t(\xi)$ is dense in T_R and since $t(\xi)$ commutes with $m(\xi)$ so does T_R.

Let $\pi : R \to M_R$ be the restriction of the projection mapping and let

$$\pi^* : R/D \cap M_R \to M_R/D \cap M_R$$

be the homeomorphism induced by π. Since T_R acts trivially on $m(\xi)$ it is obvious that

$$\pi^* \circ p(\xi) = m(\xi) \circ \pi^*, \qquad \xi \in \boldsymbol{R},$$

and we have proved our theorem.

Theorem 4.1 is a special case of the equivalence theorem that we will ultimately prove in this section. We have presented the special case first because, for class R solvmanifolds, certain constructions that we will need in the general case are so natural as to almost become trivial and disappear. However by going back and forth between the class R case and the general case the reader will be able to see the reason for construction that at first sight may seem artificial.

Let R be a connected, simply connected solvable Lie group with D a closed cocompact subgroup. Let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R and $D_S = M_R \rtimes T_D$, $T_D \subset T_R$ be the semisimple splitting of D. Let $\mathscr{A}(M_R)$ denote the automorphism group of M_R and let $A_h(T_D)_0$ denote the identity component of $A_h(T_D)$, the algebraic hull of T_D in $\mathscr{A}(M_R)$. Let $\rho : D_S \to T_D$ be the homomorphism with kernel M_R and let ρ^* be the restriction of ρ to D. Let $D_1 = p^{*-1}(T_D \cap A_h(T_D)_0)$. Since $A_h(T_D)$ has only a finite number of components D/D_1 is finite. We will call D_1 the pre-divisible subgroup of D and if $D = D_1$ we will call D itself pre-divisible.

Now pre-divisible groups are important because of the following result.

THEOREM 4.2. *Let R/D be a presentation of a compact solvmanifold, where D is a pre-divisible group. Then there exists a connected, simply connected solvable Lie group G such that*

1. *$G \supset D$ and G/D is compact.*
2. *If $G_S = M_G \rtimes T_G$ then $M_G = M_R$ and $T_G \subset A_h(T_D)_0$.*
3. *$R \subset G \rtimes C$, where C is a compact abelian group and $D \subset G \cap R$.*
4. *D_0 is normal in G.*
5. *The projection mapping $\pi^* : G \rtimes C \to G$ restricted to R is a homeomorphism and π^* induces a homeomorphism*

$$\pi^* : R/D \to G/D.$$

PROOF. Consider $R_S = M_R \rtimes T_R$ and $D_S = M_R \rtimes T_D$. Then $T_D \subset T_R$, and T_R/T_D is compact. Now $A_h(T_D)$ is connected since D is pre-divisible. Since anything that commutes with T_D commutes with $A_h(T_D)$, we have T_D and $A_h(T_D)$ form a commutative group. Because T_R/T_D is compact, if $A = T_R A_h(T_D)$,

$$A = C \oplus W$$

where W is a vector group and C is compact and connected. This direct sum representation of A is not unique, but since T_D can have no elements of finite order $T_D \cap C$ is the identity and so we may choose W so that $T_D \subset W$. Let V be the linear hull of T_D in W. Then we have

$$T_R \subset C \oplus V \quad \text{and} \quad T_D \subset V.$$

Let λ be the projection homomorphism of $C \oplus V \to V$ restricted to T_R.

Now consider $M_R \rtimes (C \oplus V)$. Let N as usual be the nil-radical of R. Then N is normal in $M_R \rtimes (C \oplus V)$ and let

$$\beta : M_R \rtimes (C \oplus V) \to (M_R/N) \oplus (C \oplus V)$$

be the homomorphism with kernel N. Consider the subgroup Δ of $(M_R/N) \oplus (C \oplus V)$ defined as follows: For $r \in R/N$ let $m^*(r)$ be the image of r in M_R/N and $t^*(r)$ the image of r in $(C \oplus V)$ relative to the direct sum decomposition $(M_R/N) \oplus (C \oplus V) \supset R/N$. Let Δ consist of all pairs

$$(m^*(r), \lambda t^*(r)) \subset (M_R/N) \oplus (C \oplus V), \qquad r \in R/N.$$

Define G as the pre-image of Δ in $M_R \rtimes (C \oplus V)$ under the mapping β. It remains to verify that G satisfies conditions 1 through 5 above. Conditions 1 and 2 are trivially true by construction. Since Δ is normal in $M_R/N \oplus (C \oplus V)$, we have G is normal in $M_R \rtimes (C \oplus V)$ and so we may form $C \ltimes G$. The rest of assertion 3 is now obvious. To verify 4, note that D_0 is normal in $M_R \rtimes A_h(T_D) \supset G$. Condition 5 is also obvious by construction since $T_D \subset V$.

DEFINITION. Let D be a pre-divisible group and let G be as construction in Theorem 4.2. We will call G a *divisible envelope* of D.

We have now come to our main theorem of this section. We will often refer to Theorem 4.3 below as the Equivalence Theorem for G-induced flows.

THEOREM 4.3. *Let R/D be a presentation of a compact solvmanifold and let $p(\xi)$ be a G-induced flow in general position in R/D. Let $D_1 \subset D$ be a pre-divisible subgroup of D and let $\Gamma = D_1/(D_1)_0$. Further let G be a divisible envelope of D_1 and let $S = G/D_0$. Then there exists a one parameter subgroup $q(\xi)$ in general position in S such that the G-induced flow induced by $q(\xi)$ in S/Γ is equivalent to the G-induced flow induced by $p(\xi)$ on R/D_1.*

(The proof of Theorem 4.3 is essentially the same as that of Theorem 4.1.)

PROOF. Choose an appropriate semidirect product representation $C \ltimes G$ so that if

$$p(\xi) = c(\xi)g(\xi), \qquad c(\xi) \subset C, g(\xi) \subset G,$$

then $c(\xi)$ and $g(\lambda)$, $\xi, \lambda \in \boldsymbol{R}$, commute and both are one parameter subgroups. This can be done because $p(\xi)$ being in general position implies that its image in C under the homomorphism $C \ltimes G \to C$ with kernel G is dense.

Because D_0 is normal in G we may let

$$\gamma : G \to G/D_0$$

and let $q(\xi) = \gamma(g(\xi))$. The rest of the proof goes by way of the usual commutative diagram.

DEFINITION. Let all notation be as in Theorem 4.3. We will call the one parameter subgroup $q(\xi)$ the *image* of the one parameter group $p(\xi)$.

5. ***G*-induced flows on class *R* solvmanifolds.** Before beginning the general study of G-induced flows on compact solvmanifolds we will first study the special case of G-induced flows on compact class R solvmanifolds. The reason for doing this is that the generalized Mauntner lemma of the next section will enable us to reduce our problem to this special case. Let us begin this section by reviewing some facts that we mentioned when we discussed nilflows.

Let X and Y be compact manifolds, let $\pi : X \to Y$ be a covering mapping, and let $x(\xi)$ be a flow on X and $y(\xi)$ a flow on Y such that $p \circ x(\xi) = y(\xi) \circ p$, $\xi \in \boldsymbol{R}$. Then $x(\xi)$ is distal (minimal) if and only if $y(\xi)$ is distal (minimal).

These general facts can be combined with Theorem 4.1 to prove the following result.

THEOREM 5.1. *Let $p(\xi)$ be a G-induced flow in general position in the presentation R/D of a compact class R solvmanifold. Then the G-induced flow is distal. Further, let $D_1 \subset R$ be such that $D/D_1 \cap D$ is finite and $D_1/D_1 \cap D$ is finite. Then $p(\xi)$ acts ergodically on R/D if and only if it acts ergodically on R/D_1.*

PROOF. Let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R. Then, by Theorem 3.5, $R/D \cap M_R$ is homeomorphic to $M_R/D \cap M_R$, a compact nilmanifold, and by Theorem 4.1 the flow induced by $p(\xi)$ on $R/D \cap M_R$ is equivalent to a nilflow on $M_R/D \cap M_R$. Hence, because R/D is a finite covering of $R/D \cap M_R$ and $p(\xi)$ acts distally on $R/D \cap M_R$, it must act distally on R/D.

We will now show $p(\xi)$ acts ergodically on R/D if and only if it acts ergodically on $R/D \cap M_R$. This fact combined with known facts about nilflows will imply the second assertion of our theorem.

Since $R/D \cap M_R$ is finitely covered by R/D, if $p(\xi)$ does not act ergodically on R/D it cannot act ergodically on $R/D \cap M_R$.

If $p(\xi)$ does not act ergodically on $R/D \cap M_R$ it has no dense orbit. This is because one dense orbit would imply, since the flow is distal, that the flow is minimal. But a minimal distal nilflow is ergodic which we assumed to be false. Hence $p(\xi)$ acting on R/D is not minimal but is distal. Hence $p(\xi)$ cannot have one dense orbit. Hence $p(\xi)$ cannot operate ergodically on R/D for if $p(\xi)$ acts ergodically on R/D it has at least one dense orbit.

Let us now introduce the concept of an unstable ideal or for our purposes, unstable normal subgroup, from which we will derive in this section a condition on a G-induced flow on a presentation of a compact solvmanifold that assures us that the solvmanifold is class R.

Because the concept of unstable ideal introduced in [**2**] seems to be well understood, I will content myself with outlining the definition and results from [**2**] and making a few remarks.

Let $\mathscr{L}$ be a Lie algebra and let $X \in \mathscr{L}$, $X \neq 0$. Decompose $\mathscr{L}$ as a vector space into its primary components relative to $\operatorname{ad} X$,

$$\mathscr{L} = \mathscr{L}_0 \oplus \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_m.$$

Here each $\mathscr{L}_i$ is invariant under ad X, and the minimal polynomial of the restriction of ad X to $\mathscr{L}_i$ is an irreducible polynomial $\mu_i(\lambda)$, $\mathscr{L}_i$ is the set of elements Z in $\mathscr{L}$ such that $\mu_i(\operatorname{ad} X)^r Z = 0$ for some r. We will call $\mathscr{L}_i$ an *unstable component* if the roots of $\mu_i(\lambda) = 0$ have nonvanishing real parts. The smallest subalgebra containing all the unstable components is an ideal which we will call the unstable ideal relative to X.

If $p(\xi)$ is a one parameter subgroup of R, we will define the unstable normal subgroup U relative to $p(\xi)$ by passing to the Lie algebra of R, $L(R)$, and, letting X denote the tangent vector to $p(\xi)$ at the identity, define U as the connected subgroup of R whose Lie algebra is the unstable ideal relative to X. It is a simple consequence of the Jordan canonical form theorem for matrices that $p(\xi)$ and its semisimple part $t(\xi)$ have the same unstable normal subgroup.

We now come to a theorem that gives a condition on a G-induced flow on a presentation R/D that forces R to be class R. That this can be done is interesting in its own right and very important for our later considerations.

THEOREM 5.2. *Let R/C be a presentation of a compact solvmanifold and let $p(\xi)$ be a one parameter subgroup in general position in R/C. If the unstable normal subgroup U of $p(\xi)$ is trivial then R is class R.*

PROOF. By Theorem 3.6 all we have to prove is that D/D_0, the fundamental group of R/D, contains a nilpotent subgroup of finite index. We then may, replacing D by a subgroup of finite index if necessary, assume that D is a pre-divisible group and we may apply Theorem 4.3. Thus we may replace R/D by S/Γ, $p(\xi)$ by $q(\xi)$, where $q(\xi)$ is in general position in S/Γ. It is straightforward to verify that the unstable normal subgroup of $q(\xi)$ in S is also trivial.

We will want one further simplification of our problem and this results from two facts:

Let $S_S = M_S \rtimes T_S$ be the semisimple splitting of S.
Then

1. S_S is class R if and only if $M_S/[M_S, M_S] \rtimes T_S$ is class R.

2. If Γ is a discrete cocompact subgroup of Γ then $\Gamma \cap [M_S, M_S]$ is discrete and cocompact. The first fact is easily verified, the second is a simple corollary of the discrete semisimple splitting theorem in [**4**].

If we let $\eta : S \to S/[M_S, M_S] = S^*$ be the natural homomorphism and let $\Gamma^* = \eta(\Gamma)$ and $q^*(\xi) = \eta(q(\xi))$. Then we have S^*, Γ^* and $q^*(\xi)$ satisfy the hypothesis of our theorem and if we can prove that S^* is class R we have proven our theorem.

Now because the nilshadow of S^* is abelian we have

$$S^* = \boldsymbol{Z}^b \otimes \boldsymbol{R} \ltimes \boldsymbol{Z}^a \otimes \boldsymbol{R},$$

$$\Gamma^* = \boldsymbol{Z}^b \otimes 1 \ltimes \boldsymbol{Z}^a \otimes 1.$$

Hence the adjoint mapping maps Γ^* into $\boldsymbol{C}^a$. Further, since the unstable ideal of $q^*(\xi)$ is trivial, $t^*(\xi)$, the semisimple part of $q^*(\xi)$, is contained in $(\boldsymbol{C}^*)^a$, where $\boldsymbol{C}^*$ denotes the multiplicative group of complex numbers of absolute value one.

We will now show that there exists at least one $z \in \boldsymbol{Z}^b \otimes 1$ such that $\operatorname{ad}(z) \in (\boldsymbol{C}^*)^a$. Note that $\operatorname{ad} z$ is an integer unimodular matrix relative to $\boldsymbol{Z}^a \otimes 1$. Since an integer unimodular matrix all of whose eigenvalues have norm sufficiently close to one has the property that its eigenvalues all have norm one, and since $q^*(\xi)$ gets arbitrarily close to some element of $\boldsymbol{Z}^b \otimes 1$, we have our first assertion.

Now let $z_1 \in \boldsymbol{Z}^b$ be such that $\operatorname{ad}(z_1) \in (\boldsymbol{C}^*)^a$. Let $z_2, \ldots, z_b$ be such that $z_1, z_2, \ldots, z_b$ generate a subgroup of finite index in $\boldsymbol{Z}^b$. Let

$$\boldsymbol{Z}^{b-1} \otimes \boldsymbol{R} \subset \boldsymbol{Z}^b \otimes \boldsymbol{R}$$

be generated by $z_2 \otimes 1, \ldots, z_b \otimes 1$. Then if we let

$$S_1 = \boldsymbol{Z}^{b-1} \otimes \boldsymbol{R} \ltimes \boldsymbol{Z}^a \otimes \boldsymbol{R}$$

and let $\Gamma_1 = \Gamma \cap S_1$ we have S_1/Γ_1 is compact.

Clearly $S = \boldsymbol{Z}^1 \otimes \boldsymbol{R} \ltimes S_1$ and we may, letting $q^*(\xi) = q_1^*(\xi)q_2^*(\xi)$, $q_1^*(\xi) \subset \boldsymbol{Z}^1 \otimes \boldsymbol{R}$ and $q_2^*(\xi) \subset S_1$, choose one representation of the semi-direct product so that $q_1^*(\xi)$ and $q_2^*(\xi)$ commute and are each one parameter groups. A simple induction on dimension of S then proves our assertion.

6. ***G*-induced ergodic flows on compact solvmanifolds.** Theorems 5.2 and 4.1 together provide a necessary and sufficient condition for a G-induced flow on a compact solvmanifold, with trivial unstable normal subgroup to act ergodically. We will now see how using the generalized Mauntner lemma we can reduce the general case to the case with trivial unstable normal subgroup. Most of this reduction already appears in [**2**] and we will content ourselves with recalling, without proof, the following results from [**2**].

THEOREM 6.1. *Let $g \to U(g)$ be a continuous unitary representation of a connected Lie group whose Lie algebra is $\mathscr{L}$. If $U(\exp \xi X)\psi = e^{i\lambda\xi}\psi$, some $\lambda \in \boldsymbol{C}$ and $\xi \in \boldsymbol{R}$, then $U(\exp Y)\psi = \psi$ for all Y in the unstable ideal relative to X.*

THEOREM 6.2. *Let $p(\xi)$ be a G-induced flow on R/D and let U be the unstable normal subgroup relative to $p(\xi)$. Let*

$$\phi : R/D \to R/(DU)^-$$

be the canonical mapping, where the bar denotes the closure operator. Then every eigenvector of $p(\xi)$ acting on $L^2(R/D)$ is of the form $\psi \circ \phi$, where ψ is an eigenvector of $p(\xi)$ acting on $L^2(R/(DU)^-)$.

Theorem 6.2 has the following immediate corollary.

COROLLARY 6.3. *Let $p(\xi)$ be a G-induced flow for the presentation R/D of a compact solvmanifold and let U be the unstable normal subgroup relative to $p(\xi)$. Further, let H be the maximal analytic subgroup of $(DU)^-$ normal in R and let $G = R/H$, $D^* = (DU)^-/H$ and $q(\xi)$ be the image of $p(\xi)$ in G. Then $p(\xi)$ acts ergodically on R/D if and only if $q(\xi)$ acts ergodically on G/D^*.*

We will now see that if $p(\xi)$ is in general position in R/D then we can conclude that G in Corollary 6.3 is class R. This gives a strong indication that the sort of reduction we are using will work to solve Problem 1.

THEOREM 6.4. *Let $p(\xi)$ be a G-induced flow in general position for the presentation R/D of a compact solvmanifold and let U be the unstable normal subgroup of $p(\xi)$. Let H be the maximal connected subgroup of $(DU)^-$ normal in R. Then $G = R/H$ is class R.*

PROOF. This amounts to checking that if $q(\xi)$ denotes the image of $p(\xi)$

in $R/H = G$ then $q(\xi)$ has trivial unstable normal subgroup and so we may apply Theorem 5.2 to prove our assertion. We will leave the details to the reader.

We will now show that we may apply Theorem 4.3 in trying to solve Problem 1.

THEOREM 6.5. *Let $p(\xi)$ be a G-induced flow in general position for the presentation R/D of a compact solvmanifold. Further, let $D_1 \subset R$ be such that $D/D_1 \cap D$ and $D_1/D_1 \cap D$ are finite. Then $p(\xi)$ acts ergodically on R/D if and only if it acts ergodically on R/D_1.*

PROOF. Since the unstable normal subgroup U relative to $p(\xi)$ does not depend on D, and D and D_1 are commensurable, it follows that $((DU)^-)_0 = ((D_1U)^-)_0$. Hence the maximal connected subgroup H of $(UD)^-$ normal in R has the same property in $(UD_1)^-$. Let $D^* = (UD)^-/H$ and $D_1^* = (UD_1)^-/H$. Then D^* and D_1^* are commensurable. Let $q(\xi)$ be the image of $p(\xi)$ in $G = R/H$. Then, by Corollary 6.3, $p(\xi)$ is ergodic on R/D or R/D_1 if and only if $q(\xi)$ is ergodic on G/D^* or G/D_1^* respectively. But, by Theorem 6.4, G/D^* and G/D_1^* are commensurable class R solvmanifolds. Hence, by Theorem 5.1, $q(\xi)$ is ergodic on G/D^* if and only if it is ergodic in G/D_1^*. This proves our assertion.

As an immediate corollary of Theorems 6.5 and 4.3 we have

COROLLARY 6.6. *Let R/D be a presentation of a compact solvmanifold and let $p(\xi)$ be a G-induced flow in general position in R/D. Now let D^* be a pre-divisible subgroup of D and let G be a divisible envelope for D^*. Then $p(\xi)$ is ergodic on R/D if and only if the induced flow $q(\xi)$ is ergodic on G/D^*.*

It is important to note, but simple to verify, the following three facts:

1. $p(\xi)$ and $q(\xi)$ have the same unipotent part in M_R.
2. $p(\xi)$ and $q(\xi)$ have the same unstable normal subgroup in M_R.
3. $q(\xi)$ is in general position in G/D^*.

We now come to the theorem that explains why divisible envelope solvmanifolds, i.e., solvmanifolds G/D where D is a pre-divisible group and G is a divisible envelope of D, are better to work with than ordinary solvmanifolds.

THEOREM 6.7. *Let G/D^* be a presentation of a compact divisible envelope solvmanifold. Further, let I be any connected normal subgroup of G. Then $(D^*I)_0^-$ is normal in G.*

PROOF. We will actually prove that $(D^*I)_0^-$ is normal in G_S and hence, since $G_S \supset G$, in G. We first observe that since $(DI)_0^-$ is invariant under inner automorphism by elements of D^* it is invariant under D_S^*. Hence $(DI)_0^-$ is invariant under M_G, the nilshadow of G. Similarly $(D^*I)_0^-$ is invariant under T_{D^*}, the semisimple part of D^*. Now T_{D^*} is Zarisky dense in its algebraic hull, $A_h(T_D)$, and so $(D^*I)_0^-$ is invariant under $A_h(T_D)$. But $A_h(T_D) \supset T_G$ and hence $(D^*I)_0^-$ is invariant under T_G. This proves that $(D^*I)_0^-$ is normal in G_S.

Let us now see where we stand after all these reductions have been made. We begin with a one parameter subgroup $p(\xi)$ in general position in the presentation of the compact solvmanifold R/D and ask for a necessary and sufficient condition for the G-induced flow to be ergodic. We replace D by a pre-divisible subgroup D^*, R by a divisible envelope of G of D^* and $p(\xi)$ by an induced flow $p^*(\xi)$ with the same unipotent part as $p(\xi)$. We next observe that if we can find a necessary and sufficient condition for $p^*(\xi)$ to act ergodically on G/D^* this will be a necessary and sufficient condition for $p(\xi)$ to act ergodically on R/D. Theorem 6.7 then tells us that if U is the unstable normal subgroup of $p^*(\xi)$ acting on G then $(UD^*)_0^- = H$ is normal in G. Hence we may replace G by $S = G/H$, D^* by $D^*H/H = \Gamma_1$ and $p^*(\xi)$ by its image $q(\xi)$ in S. Now Γ_1 is discrete in S and S is class R. We may now apply Theorem 4.1 to resolve Problem 1.

This is essentially how we will solve Problem 1. However we want our solution to be in terms that are natural. We will therefore introduce a few more algebraic constructions.

Let R/D be the presentation of a compact solvmanifold and let $\Gamma = D/D_0$. We will define the semisimple splitting of Γ, Γ_S, as follows:

Let the semisimple splittings of D, $D_S = M_R \rtimes T_D$ and let $M = M_R/D_0$ and let T_Γ be the automorphisms that T_D induce on M. Note, since T_D is discrete, the homomorphism $D \to T_D$ factors through $D \to D/D_0$. We now define $\Gamma_S = M_\Gamma \rtimes T_\Gamma$ (for an intrinsic definition of Γ_S see [**4**]). We will call M_Γ the nilshadow of Γ and T_Γ the semisimple part of Γ. We know from [**4**] that M_Γ has a unique rational form determined by Γ. We will denote this rational form of M_Γ by $(M_\Gamma)_Q$.

We now define in M_Γ the Γ unstable normal subgroup U_Γ^* as the normal subgroup of M_Γ generated by the unstable subgroup relative to $\gamma \in \Gamma$. Let U_Γ be the closure of U_Γ^* in the rational structure of M_Γ. Let $\mu: M_R \to M_\Gamma/U_\Gamma$ be the composite of the homomorphism $M_R \to M_R/D_0 = M_\Gamma \to M_\Gamma/U_\Gamma$. Then $u(m(\xi))$, $m(\xi)$ a one parameter subgroup in M_R, is called the image of $m(\xi)$ in M_Γ/U_Γ.

We may now state our solution to Problem 1.

THEOREM A. *Let R/D be a presentation of a compact solvmanifold, $p(\xi)$ a one parameter subgroup in general position in R with unipotent part $m(\xi)$. Then $p(\xi)$ acts ergodically on R/D if and only if the image of $m(\xi)$ in the rational nilpotent group M_Γ/U_Γ is in general position.*

REMARK. The rational structure on M_Γ/U_Γ depends only on Γ.

PROOF. By the discussion following the proof of Theorem 6.7, we see that the proof of Theorem A reduces to proving that $M_R/H = M_\Gamma/U_\Gamma$ or, what amounts to the same thing, that $(M_\Gamma/D_0)/(H/D_0) = M_\Gamma/U_\Gamma$. This amounts to showing that if $q(\xi)$ is in general position in S/Γ_1 then the rational closure of the unstable ideal of $q(\xi)$ is the rational closure of one unstable ideal of Γ_1. But this is the same as showing the two unstable ideals are equal.

Since M_R/H is class R, it follows that $U \supset U_\Gamma$. But since U_Γ is maximal with respect to the property of containing all the eigenspaces with eigenvalues not of absolute value one, it follows that $U_\Gamma \supset U$ and so $U_\Gamma = U$ and we have proven our theorem.

7. **Compact solvmanifolds with G-induced flows.** Until now in presenting details of this paper, although we have occasionally borrowed details from [**4**], we have developed the way of thinking about our problems internally to this paper. However this is no longer possible in this section. Except for the statement of theorems we will assume that the reader has an in-depth knowledge of [**4**].

Let us begin by stating Theorems B and C which are solutions to Problems 2 and 3 respectively.

THEOREM B. *Let X be a compact solvmanifold with fundamental group Γ. Let $\Gamma_S = M_\Gamma \rtimes T_\Gamma$ be the semisimple splitting of Γ and let $U_\Gamma \subset M_\Gamma$ be the rational closure of the unstable ideal of Γ. Then X has an ergodic G-induced flow if and only if $\Gamma/\Gamma \cap U_\Gamma$ is a nilpotent group.*

THEOREM C. *Let X be a compact solvmanifold with minimal G-induced flow. Then X is homeomorphic to a nilmanifold and the flow is equivalent to a nilflow.*

Since Theorems A and C readily yield one of the implications of Theorem B, we will begin with a proof of Theorem C.

PROOF OF THEOREM C. By Theorem 4.1 and the discussion in §3, the assertion of Theorem C amounts to showing that if R/D is a presentation of X with a minimal G-induced flow $p(\xi)$ then D is nilpotent. Now let $R_S = M_R \rtimes T_R$ be the semisimple splitting of R. Then, by the discrete semisimple splitting theorem of [**4**], it follows that $[M_R, M_R]/D \cap [M_R, M_R]$ is compact. Further, it is clear that D is nilpotent if and only if $D/D \cap [M_R, M_R]$ is abelian.

Now let $R^* = R/[M_R, M_R]$, $D^* = D/D \cap [M_R, M_R]$, and let $p^*(\xi)$ be the image of $p(\xi)$ in R^*. If $p(\xi)$ is minimal in X then $p^*(\xi)$ is minimal in R^*/D^*. Hence we will have proven our result once we have shown that the minimality of $p^*(\xi)$ on R^*/D^* implies that D^* is abelian.

Assume that $p^*(\xi)$ acts minimally on R^*/D^*, but that D^* is not abelian. By the Appendix to this paper, there exists nonempty connected subgroups B and A such that if N^* is the nilradical of R^*,

$$A \subset N^* \quad \text{and} \quad R^* = B \ltimes A.$$

Further, $B/B \cap D^*$ and $A/A \cap D^*$ are compact.

Since $p^*(\xi)$ acts minimally on R^*/D^*, $p^*(\xi)$ is in general position in R^*/D^*. From this it follows that there exists a one parameter subgroup $p_0^*(\xi) \subset B$ and $n \in N^*$ such that

$$p^*(\xi) = n_0 q_0^*(\xi) n_0^{-1}.$$

Now if we use rD as cosets, $r \in R^*$, the image of the set nB in R^*/D^* is compact, since $B/B \cap D^*$ is compact. Now

$$p^*(\xi)n_0 = (n_0 p_0^*(\xi) n_0^{-1})n_0 = n_0 p_0^*(\xi).$$

Hence the orbit $p^*(\xi)n_0$ is in a compact subset of R^*/D^* of dimension less than the dimension of R^*/D^*. This contradiction of $p^*(\xi)$ being minimal implies that D^* is abelian and we have proven our assertion.

We now come finally to the proof of Theorem B.

PROOF OF THEOREM B. Assume X is a compact solvmanifold with a presentation R/D with a G-induced flow that is ergodic. Let U be the rational closure of the unstable ideal of the one parameter group $p(\xi)$ whose action on R/D is ergodic and let $E = (DU)^-$. Then $p(\xi)$ acts ergodically on R/E. But $p(\xi)$ acts distally on R/E. Hence $p(\xi)$ acts minimally on R/E. Thus, by Theorem C, R/E is equivalent to a nilmanifold or the fundamental group of R/E is nilpotent. But the fundamental group of R/E is $\Gamma/\Gamma \cap U_\Gamma$.

Assume now that $\Gamma/\Gamma \cap U_\Gamma$ is nilpotent, we need to produce a presentation for X with ergodic G-induced flow. In presenting the rest of the details of this proof we will assume that the reader is familiar with the details in §§5 and 6 of Chapter III of [**4**].

Let Γ be the fundamental group of X and let $\Gamma_S = M_\Gamma \rtimes \boldsymbol{Z}^S$, where M_Γ is the connected nilshadow of Γ. Form $M_\Gamma \otimes \boldsymbol{C}$, the complex points of the nilpotent Lie group M_Γ, and extend $\boldsymbol{Z}^S$ acting on $M_\Gamma \otimes \boldsymbol{C}$ to $\boldsymbol{Z}^S \otimes \boldsymbol{R}$ acting on $M_\Gamma \otimes \boldsymbol{C}$ in such a way that all eigenvectors of eigenvalue 1 for $z \in \boldsymbol{Z}^S$ have this property also for $z \otimes \boldsymbol{R}$. The construction of §5 of Chapter III of [**4**] yields a presentation R/D of X. It is easily checked that there is a natural homomorphism of $R \to M_\Gamma/U_\Gamma$ which sends D to $\Gamma/\Gamma \cap U_\Gamma$. Choose any one parameter subgroup $p(\xi)$ in general position in R which maps onto a one parameter subgroup of M_Γ/U_Γ which is in general position relative to the rational form determined by $\Gamma/\Gamma \cap U_\Gamma$. By Theorem A it is easily seen that $p(\xi)$ has a G-induced ergodic action on R/D.

8. **Appendix.** In the proof of Theorem C in §7 we needed some detailed

facts about the structure of presentations of solvmanifolds whose fundamental groups had abelian nilshadows. Since this material does not appear to be in the literature and has considerable independent interest, I will present the details in this appendix.

Let us begin by reviewing some facts about rational matrices.

Let V^n be an n-dimensional real vector space with rational form $V^n_{\boldsymbol{Q}}$. We will call a subspace $W \subset V^n$ *rational* if $W \cap V^n_{\boldsymbol{Q}}$ is a rational form of W. Now let A be a linear transformation of V^n preserving the rational form $V^n_{\boldsymbol{Q}}$. Assume that A is semisimple over the reals; then A is semisimple over the rationals. In other words, if W_1 is a rational subspace invariant under A, there exists a rational subspace W_2 such that

(a) W_2 is A-invariant,

(b) $V^n = W_1 \oplus W_2$.

This may be seen as follows: Let $e_1, \ldots, e_n$ be a basis of $V^n_{\boldsymbol{Q}}$ such that $e_1, \ldots, e_k$ are a basis of W_1. Then A has a matrix representation

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$$

relative to $e_1, \ldots, e_n$ as column vectors. Further all matrix entries are rational numbers. Let X be a real matrix such that XAX^{-1} is of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}.$$

X exists, because A is semisimple and X has the form

$$\begin{pmatrix} I & \chi \\ 0 & I \end{pmatrix}$$

where I is the identity matrix. The entries in χ are then determined by a system of linear equations with rational coefficients. Since it has real solutions it has rational solutions.

Let us look at an application of this result. Let $L \subset V^n$ be a lattice and let $V^n_{\boldsymbol{Q}}$ be the rational form containing L. Let A be a linear transformation preserving the lattice L and assume that A is semisimple over the reals. Let W_1 be the eigenvalue 1 space of A and let W_2 be the range of $(A - I)$. Then W_1 and W_2 are rational vector spaces and $V = W_1 \oplus W_2$.

Clearly $(A - I)$ is an integer matrix leaving W_2 invariant. Hence, by previous discussion, W_1 is rational.

Finally, let T be a semisimple abelian group of matrices which are rational relative to the rational form $V^n_{\boldsymbol{Q}}$. Let ρ be the smallest linear subspace of V^n containing the range of $(t - I)$, $t \in T$. Then

(a) ρ is T-invariant,

(b) ρ is rational,

(c) there exists a rational subspace μ such that $\rho \oplus \mu = V^n$ and $T|\mu$ acts trivially.

The proof of these assertions is a trivial consequence of what has already been established.

We will henceforth call ρ the range of T and μ the common eigenvalue one space of T.

Now let N be the nilradical of R. Then $\rho_s \subset R$ and hence is an abelian has an abelian nilshadow. Let R/D be a presentation of X. We will give a fairly complete description of R and D. In presenting the details of the rest of this section we will assume the reader is familiar with [**4**].

Let $\Gamma_S = \boldsymbol{Z}^t \ltimes \boldsymbol{Z}^s$ be a discrete semisimple splitting of Γ and let $R_{\tilde{S}} = \boldsymbol{Z}^t \otimes \boldsymbol{R} \ltimes \boldsymbol{R}^\alpha$ be the semisimple splitting of R. Let $D \subset R$ be such that R/D is a presentation of X, then $\boldsymbol{R}^\alpha/D_0 = \boldsymbol{Z}^s \otimes \boldsymbol{R}$, D_0 is $\boldsymbol{Z}^t \otimes 1$ invariant, and $\boldsymbol{Z}^t$ acting on $\boldsymbol{Z}^s \otimes \boldsymbol{R}$ is induced by $\boldsymbol{Z}^t \otimes 1$ acting on $\boldsymbol{R}^\alpha$. Now, since $\boldsymbol{Z}^t$ acts semisimply on $\boldsymbol{Z}^s \otimes \boldsymbol{R}$, $\boldsymbol{Z}^t \otimes 1$ acts semisimply on $\boldsymbol{R}^\alpha$. Hence both $\boldsymbol{Z}^s \otimes \boldsymbol{R}$ and $\boldsymbol{R}^\alpha$ decompose as a direct sum of character spaces. Because D_0 contains no connected subgroup normal in R, it follows easily that every character of $\boldsymbol{Z}^t \otimes 1$ acting on $\boldsymbol{R}^\alpha$ appears in $\boldsymbol{Z}^t$ acting on $\boldsymbol{Z}^s \otimes \boldsymbol{R}$.

Now let p_α and p_s be the range of $\boldsymbol{Z}^t$ acting on $\boldsymbol{R}^\alpha$ and $\boldsymbol{Z}^s \otimes \boldsymbol{R}$, respectively, and let μ_α and μ_s be the common eigenvalue one space of $\boldsymbol{Z}^t$ acting on $\boldsymbol{R}^\alpha$ and $\boldsymbol{Z}^s \otimes \boldsymbol{R}$, respectively. Then $\rho_\alpha/\rho_\alpha \cap K_D$ and $\mu_\alpha/\mu_\alpha \cap K_D$ are compact, where K_D is the nilshadow of the discrete splitting of D.

This result may be seen as follows: Consider a character space for $\boldsymbol{Z}^t$ acting on $\boldsymbol{R}^\alpha$ and $\boldsymbol{R}^s$ and call these character spaces V_α and V_s respectively. Then $V_\alpha/V_\alpha \cap D_0 = V_s$. Hence $\rho_\alpha/D_0 \cap \rho_\alpha = \rho_s$. Since $\rho_s/(\rho_S \cap K_D/D_0)$ is compact we have that $\rho_\alpha/K_D \cap \rho_\alpha$ is compact. A similar argument holds for μ_0.

Now let N be the nilradical of R. Then $\rho_s \subset R$ and hence is an abelian normal subgroup of R. Change notation and call it A. Then $A/A \cap D$ is compact. Now let $\pi: R \to \boldsymbol{R}^\alpha$ be the restriction of the projection mapping of $R_{\tilde{S}}$ onto $\boldsymbol{R}^\alpha$ to R. Let $\pi^{-1}(\mu_\alpha) = B$. Then clearly $B/B \cap D$ is compact. It is straightforward to verify (because B is invariant under $\boldsymbol{Z}^t \otimes \boldsymbol{R}$) that B is a subgroup of R. Hence

$$R = B \ltimes A.$$

Further, since the fundamental group of B is abelian, $B/B \cap D$ is a torus and its structure is known from the work of R. W. Johnson [7]. Thus R/D is a torus bundle over a torus.

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