estimates stemming from Carleman for operators having an imaginary part lying in a p-ideal. The latter results make extensive use of the theory of analytic functions and especially of entire functions.

The principal application of these results in this volume is to the completeness problem of root spaces. Approximately one-third of the book is devoted to this problem along with a study of the various kinds of bases which can exist in Hilbert space and the notion of expansion appropriate to each. The early and fundamental results of M. V. Keldys in this area are presented in complete detail. A rather thorough presentation of this area is given with various techniques being illustrated. Lastly, various asymptotic properties of the spectrum of weakly perturbed operators are given.

In summary this book is a thorough and complete treatment along with many worthy contributions of some important but relatively neglected areas of abstract operator theory with applications to more immediate and concrete problems. We eagerly await the remaining two volumes.

RONALD DOUGLAS

Introduction to analytic number theory, by K. Chandrasekharan, Springer, 1968; *Arithmetic functions*, by K. Chandrasekharan, Springer, 1970; *Multiplicative number theory*, by Harold Davenport, Markham, 1967; *Sequences*, by H. Halberstam and K. F. Roth, Oxford University Press, 1966.

Recent years have seen an explosion in the number of books in most branches of mathematics and this is true of number theory. Most books contain little that is new, even in book form. This is the case of 8/3 of the four books in this review. They are well written and make good textbooks and pleasant reading but they are not revolutionary. The remaining 4/3 books are new in book form and we will spend most of the review on these.

We begin this review with a discussion of Chandrasekharan's *Introduction to analytic number theory*, which is a translation with some slight revisions of the author's *Einführung in die analytische Zahlentheorie* (Springer lecture notes series number 29). This book presupposes the usual knowledge of functions of a complex variable (i.e. Cauchy's theorem) but virtually no knowledge of number theory. Indeed, the book begins with the unique factorization theorem and in the early chapters moves through (among other things) congruences, the law of quadratic reciprocity and several standard arithmetical functions. The later chapters include Weyl's theorems on uniform distribution, Minkowski's convex body theorem, Dirichlet's theorem on primes in arithmetic progressions and the prime number theorem (via the Wiener Ikahara method).

Several of the theorems in the book are proved by means of convex variables and Fourier series (one of the nicest of these being Siegel's proof of Minkowski's theorem). Still there is no circle method, no functional equations, nor even any Dirichlet series extended past their half plane of convergence. Indeed, a Dirichlet series with a complex argument is not even mentioned until p. 106 (24 pages before the end of the text). In conclusion, I would say that the book is interesting and well written but it is definitely not an introduction to analytic number theory.

This brings us to the same author's introduction to analytic number theory, which he has entitled *Arithmetic functions*. About two-thirds of the book is devoted to the distribution of primes and the rest is split between the partition function and the divisor function. Chapter One presents an elementary proof of the prime number theorem. The proof chosen follows that of Wirsing, which is capable of producing an error term but only the asymptotic result is given here. It should be mentioned that the error term that can be derived from the elementary method has just been greatly improved by Diamond and Steinig (Invent. Math. **11** (1970), 199–258). They showed that

$$\pi(x) = li(x) + O\{x \exp[-c(\log x)^{1/7-\epsilon}]\} \tag{1}$$

where as usual $\pi(x)$ denotes the number of primes $\leqq x$ and $li(x)$ is the logarithmic integral of x.

In Chapter Two we encounter Riemann's zeta function, $\zeta(s)$. It is extended to the entire plane, the number of zeros in the critical strip ($0<\sigma<1$ where $s=\sigma+it$) is given and Hardy's (but not Selberg's) theorem on the number of zeros on the critical line ($\sigma=1/2$) is given. In Chapter Three we at last get the prime number theorem with an error estimate. In fact, we are given Littlewood's improvement of the older de la Vallée-Poussin result ($O\{x \exp[-c(\log x)^{1/2}]\}$ in (1)); it is the latter that is most often given in books (e.g. Davenport's). In Chapter Four we get Vinogradov's method of trigonometrical sums and Chudakov's resulting refinement of Littlewood's theorem (but we do not get the best-known present result ($O\{x \exp[-c(\log x)^{3/5-\epsilon}]\}$ in (1)) that was later derived by Vinogradov and Korobov).

Chapter Five presents Hoheisel's and Ingham's results on the differences between consecutive primes (Ingham's result is that if x is large, then there is a prime between x and $x+x^{5/8+\epsilon}$, in particular there are primes between large consecutive cubes), and Chapter Six is devoted to L-functions, their functional equations, and Siegel's theorem on their real zeros. Some generalizations of the results of earlier chapters to primes in arithmetic progressions are stated but not proved.

Chapter Seven breaks new ground and presents the circle method applied to the partition function $p(n)$ culminating with Rademacher's infinite series for $p(n)$. The last chapter, Chapter Eight, again changes the subject and presents Dirichlet's divisor problem.

In my opinion, the main value of the book lies in Chapters 1–5 on the distribution of primes. These chapters certainly provide an updating of Ingham's Cambridge Tract, *The distribution of prime numbers* (although much less so when it is combined with Titchmarsh's book, *The theory of the Riemann zeta function*). Without the corresponding work on primes in arithmetic progressions, Siegel's theorem seems isolated and, in particular, it is difficult to appreciate the ineffective nature of the result. Sieve results are not presented even though they are at present producing most of the new results in the subject—e.g. Montgomery's improvement of Ingham's theorem (the exponent 5/8 is replaced by 3/5; this appears in [9]) "by a new method". There are notes given in each chapter to further references and original sources which should prove useful.

Multiplicative number theory by Harold Davenport is based upon a one-semester course of lectures that I was priviledged to attend. The stated aim of the book is to give an account of multiplicative number theory with particular attention to the distribution of primes in arithmetic progressions and the recent related large sieve results. There are thirty short chapters which more or less follow the historical order of development. Chapters 1–6 cover Dirichlet's theorem on primes in arithmetic progression ($\pi(x; q, a)\rightarrow\infty$ as $x\rightarrow\infty$ where $\pi(x; q, a)$ is the number of primes, $p\leqq x$, such that $p\equiv a \pmod q$ and it is assumed that $(a, q)=1$). Following Dirichlet, the theorem is first proved for prime q and then, with the aid of Dirichlet's class-number formula (to show that real L-series at $s=1$ are not 0), for general q.

Chapters 7–22 are devoted to the prime number theorem and its generalization to arithmetic progressions. Beginning with Riemann's remarkable paper of 1860, we enter the complex plane and find functional equations of ζ and L-functions, product formulae for integral functions of order 1, zero free regions for ζ and L-functions and explicit formulae for functions related to $\pi(x)$ and $\pi(x; q, a)$. Unlike the zeta function, there is no proof that L-functions cannot have real zeros between 0 and 1 and, in particular, close to 1. Such real zeros profoundly affect the results on error terms for primes in arithmetic progressions and class-numbers of quadratic fields. Siegel's theorem in Chapter 21 is the best present result on how far such a zero must be from $s=1$. In this book, one can appreciate the ineffective nature of the result and how frustrating it is to know that something goes to infinity and not be able to say when it is greater than (for example) 3.

Up to this point, only individual arithmetic progressions have been considered. In Chapters 23–29, the goal is to develop the large sieve so as to consider averages of errors over several progressions in the form of a theorem of Bombieri: given $A>0$, there exists $B>0$ such that

$$(2) \qquad \sum_{q\leq X} \max_{(a,q)=1} \max_{y\leq x} \left| \psi(y; q, a) - y/\phi(q) \right| \ll x(\log x)^{-A}$$

where

$$X = x^{1/2}(\log x)^{-B} \quad \text{and} \quad \psi(y; q, a) = \sum_{p^m \leq y;\ p^m \equiv a \pmod q} \log p.$$

Since the best-known results for each progression are of the form

$$\psi(x; q, a) = x/\phi(q) + O\{x \exp[-c(\log x)^d]\} \qquad (d < 1),$$

where, to make matters worse, the constant in O even depends on q, we can see what a great improvement (2) is over simply summing the known estimates for each term. Indeed, (2) is comparable to what one gets by adding the error estimates for each progression deduced by assuming the generalized Riemann hypothesis. This is what is behind the statement that when one is dealing with averages, results formerly derived from Riemann hypotheses may now possibly be derived from the large sieve.

The first two-thirds of the book cover old established results. Although many of the results are not the best presently known (for example, *Arithmetic functions* goes much deeper into results on $\pi(x)$), they are sufficient for most purposes and are beautifully written. The last third of the book on the large sieve is also, with one lapse (and a missing x in the statement of Renyi's result on p. 137), well written. The lapse occurs on p. 151 in an incorrectly modified form of Theorem 2 which invalidates the proof of Theorem 4. The necessary modifications appear in *Mathematika* **14** (1967), 229–232, and need not be repeated here. However, one can say that the book was written too soon for its last third. The course was given in the first months of 1966. Interest had just been rekindled in the large sieve and many people were working on it. Indeed, 3 or 4 times during the term, seminars and talks on the large sieve were given and each time it was simplified. These simplifications are incorporated in the book. Naturally enough, however, the simplifications did not cease in 1966 and the result is that now the last third of the book is overly complicated. Thus the large sieve section of the book is now of interest as a well-presented intermediate version of the large sieve which stimulated many of the later improvements.[1] We will have more to say on the later improvements in the large sieve after the review of the next book.

Numerous references are given in footnotes but are not collected anywhere (except for a list of books on p. viii). Also there is no index. In spite of these shortcomings, this book is highly recommended. It should serve as an excellent source and reference for a long time to come and taken together with Davenport's earlier work on Waring's problem (*Analytic methods for Diophantine equations and Diophantine inequalities*) would make an excellent advanced course in number theory. It would have been interesting to see what the last third of the book would look like if it were revised now but due to Professor Davenport's untimely death, this is impossible. He was a good mathematician, a good lecturer and a good writer. The combination is rare.

Much of the three previous books consist of material that is well-known to experts and has often appeared in book form. By contrast, much of *Sequences* by H. Halberstam and K. F. Roth is unique in book form. This book has been out for five years and has been reviewed several times before. However, another review may not be amiss since the book is still not well known and, besides, it deserves a review of more than its table of contents. *Sequences* is concerned with properties of wide classes of sequences of integers. Occasional special sequences occur but mostly in either existence proofs or examples of how near the general theorems come to the best results for particular sequences.

The book contains a large amount of material; to indicate how much and because some of the results are not well known, I will give the actual statements of some of highlights in the book as opposed to just names of theorems. Hence some notation is in order. This subject abounds with notation; in spite of valiant efforts by the authors, notation is still the greatest obstacle to easily reading the book. Fortunately, we will not need much of it here. All the results given below occur in the book, sometimes in other forms.

[1] Professor Halberstam has pointed out to me that *Multiplication number theory* appears in a lecture series, a proper aim of which can be to stimulate further improvements rather than give things the ultimate shape.

Let $\mathcal{A}$ and $\mathcal{B}$ denote strictly monotonic increasing sequences of nonnegative integers, either finite or infinite. $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are the sequences of integers $\geqq 0$ and $\geqq 1$, respectively. The numbers $A(n)$ and $B(n)$ are the number of *positive* elements of $\mathcal{A}$ and $\mathcal{B}$ not exceeding n. Chapter One studies densities of sums of sequences. The sum of $\mathcal{A}$ and $\mathcal{B}$ is the sequence, denoted by $\mathcal{A}+\mathcal{B}$, made up of all distinct numbers of the form $a+b$ with $a\in\mathcal{A}$, $b\in\mathcal{B}$. By $h\mathcal{A}$, we mean $\mathcal{A}+\mathcal{A}+\cdots+\mathcal{A}$ ($h-1$ plus signs). Two types of densities are appropriate here. The first is the *Schnirelmann density* of a sequence

$$\sigma\mathcal{A} = \inf_{n>0} \frac{A(n)}{n}$$

and the second is the (*lower* and *upper*) *asymptotic density*

$$d_L\mathcal{A} = \liminf_{n\to\infty} \frac{A(n)}{n}, \qquad \bar{d}\mathcal{A} = \limsup_{n\to\infty} \frac{A(n)}{n},$$

$$d\mathcal{A} = \lim_{n\to\infty} \frac{A(n)}{n}.$$

(d_L, and δ_L later, differs from the notation in *Sequences* due to typesetting restrictions of the American Mathematical Society.) The great usefulness of the Schnirelmann density is that $\sigma\mathcal{A}=1$ if and only if $\mathcal{A}=\mathcal{Z}_0$ or $\mathcal{Z}_1$.

The first results of Chapter One are those of Schnirelmann:

THEOREM. *If* $1\in\mathcal{A}$, $0\in\mathcal{B}$, *then* $\sigma(\mathcal{A}+\mathcal{B})\geqq\sigma\mathcal{A}+\sigma\mathcal{B}-\sigma\mathcal{A}\sigma\mathcal{B}$.
If $0\in\mathcal{A}$, $0\in\mathcal{B}$ *and* $\sigma\mathcal{A}+\sigma\mathcal{B}\geqq 1$, *then* $\mathcal{A}+\mathcal{B}=\mathcal{Z}_0$.

Let $\mathcal{P}$ denote the sequence of 0, 1 and the prime numbers. Schnirelmann showed by a sieve argument that $\sigma(2\mathcal{P})>0$ and then applied this theorem to show that $h\mathcal{P}=\mathcal{Z}_0$ for some h. Most of the rest of the chapter concerns improvements of this theorem. However, as it stands, the first part of the theorem is already best possible. The authors mention this as an open question on p. 4, but several people have written the authors that the example on p. 6 (used for another purpose), $\mathcal{A}=\{1, 10, 11, 12, \cdots\}$, $\mathcal{B}=\{0, 1, 9, 10, 11, 12, \cdots\}$ settles the question.

As the second half of the theorem shows, if $0\in\mathcal{A}$, $0\in\mathcal{B}$, then a result can sometimes be derived without a $\sigma\mathcal{A}\sigma\mathcal{B}$ term. To do this in general turned out to be surprisingly difficult but was ultimately done by Mann (for which he was awarded the Cole prize of the American Mathematical Society in 1946):

MANN'S THEOREM. *If* $0\in\mathfrak{A}$, $0\in\mathfrak{B}$, *then* $\sigma(\mathfrak{A}+\mathfrak{B})\geqq\min(1,\sigma\mathfrak{A}+\sigma\mathfrak{B})$.

Three years later, Dyson obtained an improvement of Mann's Theorem.

DYSON'S THEOREM. *If* $\mathfrak{A}_j$ $(j=0, 1, \cdots, k)$ *are all sequences containing* 0 and $\mathfrak{B}=\mathfrak{A}_0+\cdots+\mathfrak{A}_k$, *then for any* $n>0$ *and any* m, $0<m\leqq n$,

$$\frac{B(m)}{m}\geqq\min\left(1,\ \min_{1\leq r\leq n}\ \sum_{j=0}^{k}\frac{A_j(r)}{r}\right)\cdot$$

This immediately implies that $\sigma(\mathfrak{B})\geqq\min(1, \sum_{j=0}^{k}\sigma\mathfrak{A}_j)$. Mann's paper included the case $k=1$.

The second half of Chapter One deals with asymptotic analogues of the first half. We see from an example that the asymptotic analogue of Mann's (or Dyson's) Theorem is false. Set $\mathfrak{A}=\{0, g, 2g, 3g, \cdots\}$. Then for any k, $(k+1)\mathfrak{A}=\mathfrak{A}$ and hence

$$d((k+1)\mathfrak{A}) = (k+1)d\mathfrak{A}-\frac{k}{g}\quad(<1).$$

The main result of the second half of the chapter is Kneser's Theorem which says that the only counterexamples to the asymptotic analogue are "essentially" sequences of congruence classes.

KNESER'S THEOREM. *Let* $\mathfrak{A}_j$ $(j=0, \cdots, k)$ *be sequences and* $\mathfrak{B}=\sum_{j=0}^{k}\mathfrak{A}_j$. *Suppose that*

$$d_L\mathfrak{B}<\min\left(1, \sum_{j=0}^{k}d_L\mathfrak{A}_j\right)\cdot$$

Then there exist sequences $\mathfrak{A}_j'\supseteq\mathfrak{A}_j$ $(j\geqq 0, \cdots, k)$ *and an integer* g *such that* $\mathfrak{B}'=\sum_{j=0}^{k}\mathfrak{A}_j'$ *coincides with* $\mathfrak{B}$ *from some point onward and each* $\mathfrak{A}_j'$ *is the union of* (*the nonnegative parts of*) *congruence classes* (mod g). *Further*, g *may be so chosen that*

$$d\mathfrak{B}=d\mathfrak{B}'\geqq\sum_{j=0}^{k}d\mathfrak{A}_j'-k/g.$$

One could not ask for a nicer result than this. The readers interested in the theorems of Mann, Dyson and Kneser are referred to *Addition theorems* by H. B. Mann (Interscience Publishers, 1965) which overlaps Chapter One to a great extent.

Chapter One concerns itself with many other questions of which we will mention one here (also to be found in Mann's book). An *essential* *component* is a sequence $\mathcal{B}$ such that $\sigma(\mathcal{A}+\mathcal{B})>\sigma\mathcal{A}$ for all $\mathcal{A}$ with $0<\sigma\mathcal{A}<1$. A sequence $\mathcal{B}$ is said to be a *basis* of *order h* if $h\mathcal{B}=Z_0$. By Schnirelmann's theorem, any sequence of positive density containing 0 is both a basis and an essential component. There are bases of density zero (we have already mentioned the sequence of primes plus 0 and 1), an example is the sequence $\mathcal{Q}$ of squares (including 0) which is a basis of order 4. Khintchin proved that it is also an essential component and even

$$\sigma(\mathcal{A}+\mathcal{Q}) \geqq \sigma\mathcal{A} + 5\cdot 10^{-9}(1-\sigma\mathcal{A})^2\sigma\mathcal{A} \quad \text{for all } \mathcal{A}.$$

However, we have the later amazing result of Erdös:

THEOREM. *If $\mathcal{B}$ is a basis of order h, then for all $\mathcal{A}$,*

$$\sigma(\mathcal{A}+\mathcal{B}) \geqq \sigma\mathcal{A} + \frac{1}{2h}(1-\sigma\mathcal{A})\sigma\mathcal{A}.$$

In particular, $\mathcal{B}$ is an essential component.

Not only is this result completely general, but it even gives better numerical results than a special case previously proved by means adapted to the special case. Such results are one of the main goals of the theory of sequences but they are very rare. There is one other result in *Sequences* that is equally striking; it occurs in Chapter Two.

Chapter Two is a relatively short chapter which studies the number of representations of integers by $h\mathcal{A}$ for fixed h (usually 2). Let $R_n(\mathcal{A})$ denote the number of representations of n as $a+a'$ with a and a' in $\mathcal{A}$. We recall the well-known result of Hardy and Landau that for the sequence $\mathcal{Q}$ of squares (including 0),

$$\sum_{n=0}^{N} R_n(\mathcal{Q}) = \frac{\pi}{4}N + o(N^{1/4}(\log N)^{1/4}) \qquad \text{as } N\to\infty$$

is false. This proved by analytic techniques that depend heavily on the sequence $\mathcal{Q}$. Yet in Chapter Two we find the following theorem valid for any sequence $\mathcal{A}$.

ERDÖS-FUCHS THEOREM. *The relation*

$$\sum_{n=0}^{N} R_n(\mathcal{A}) = cN + o(N^{1/4}(\log N)^{-1/2}) \qquad \textit{as } N\to\infty$$

cannot hold for any constant $c>0$.

This theorem is certainly the highpoint of Chapter Two. We also find in a footnote on p. 106 that somewhere before 1961, W. Jurkat has proved the same result with the larger error term $o(N^{1/4})$. Unfortunately, a decade later no proof has yet appeared. There is also a multiplicative analogue of this theorem due to Richert which is stated (but not proved) on p. 106. Finally, it should be mentioned that R. Vaughan has just proved the analogue of the Erdös-Fuchs Theorem for representations by $h\mathcal{A}$ instead of $2\mathcal{A}$.

Chapter Three deals with probabilistic methods. Much of the chapter is devoted to developing the necessary probability theory. The theory is actually developed to prove four theorems (all related to Chapters One and Two) but it can certainly be applied to other problems. It is a familiar technique in number theory to show that something happens by finding an asymptotic formula for the number of times it happens. The idea of Chapter Three is to show that certain types of sequences exist by showing that the probability of choosing such a sequence "at random" is greater than 0. In fact, in each case the probability that a random sequence has the desired property turns out to be 1.

The problems under consideration are all of whether or not certain types of sequences with given growth rates exist. Probabilistic methods work because of the following two theorems. We let Ω denote the set of all sequences $\mathcal{A}\subseteq Z_1$.

THEOREM. *Let $\alpha_1, \alpha_2, \cdots$ be real numbers, $0\leqq\alpha_j\leqq 1$. There exists a probability measure μ on Ω ($\mu(\Omega)=1$) such that*

(i) *for every natural number n, the set ("event") $B^{(n)}=\{\mathcal{A}\in\Omega \mid n\in\mathcal{A}\}$ is measurable and $\mu(B^{(n)})=\alpha_n$,*

(ii) *the events $B^{(1)}, B^{(2)}, \cdots$ are independent.*

As an application of the law of large numbers, we then get the

THEOREM. *Suppose that for $j\geqq j_0$, the α_j of the last theorem are given by*

$$\alpha_j = \alpha\frac{(\log j)^{c'}}{j^c}$$

where $\alpha>0$, $c'\geqq 0$, $0<c\leqq 1$. Then with probability 1, *a sequence $\mathcal{A}$ satisfies*

$$A(n) \sim \frac{\alpha}{1-c}(\log n)^{c'}n^{1-c}, \qquad c\neq 1,$$

$$\sim \frac{\alpha}{c'+1}(\log n)^{c'+1}, \qquad c=1.$$

Thus with a suitable choice of the α_j, almost all sequences in Ω will have the same prescribed growth rate.

The most difficult of the chapter's four main theorems is the result of Erdös and Renyi:

THEOREM. *Let* $r_n(\mathfrak{A})$ *denote the number of representations of* n *as* $a+a'$ *with* a *and* a' *in* $\mathfrak{A}$, $a<a'$. *Further, let the* α_j *of the last two theorems be given by* $\alpha_j=\frac{1}{2}j^{-1/2}$, *and let* $\lambda=\pi/8$. *Then with probability* 1, *a sequence* $\mathfrak{A}=\{a_1, a_2, \cdots\}$ *satisfies the following four conditions:*

(i) $a_j\sim j^2$ *as* $j\to\infty$.

(ii) *For all* $k\geqq 0$, *the sequence* $\mathfrak{B}_k$ *consisting of those* n *with* $r_n(\mathfrak{A})=k$ *has density* $d\mathfrak{B}_k=\lambda^k e^{-\lambda}/k!$.

(iii) $\sum_{n=1}^N r_n(\mathfrak{A})\sim\lambda N$ *as* $N\to\infty$.

(iv) $$\limsup_{n\to\infty}\frac{r_n(\mathfrak{A})\log\log n}{\log n}=1.$$

The sequence of squares belongs to the complementary set of measure zero ((ii) and (iv) fail). Part (i) follows from the previous theorem but the other three parts are more difficult. The reason is that, unlike what happens in the three other main theorems of the chapter, the events that occur here are not independent. For example, one finds that

$$\mu\{\mathfrak{A}\mid r_n(\mathfrak{A})=0\}=\prod_{1\leq k<n/2}(1-\alpha_k\alpha_{n-k})\sim e^{-\lambda}\qquad\text{as } n\to\infty.$$

If for a function f defined on Ω, we let the first moment of f be

$$M(f)=\int_\Omega f\,d\mu,$$

then $B_0(N)$ (the counting number of the sequence $\mathfrak{B}_0$ in (ii)) is a function on Ω and its moment satisfies

$$\lim_{N\to\infty}M(N^{-1}B_0(N))=\lim_{N\to\infty}N^{-1}\sum_{n=1}^N\mu\{\mathfrak{A}\mid r_n(\mathfrak{A})=0\}=e^{-\lambda}.$$

The connection between this and (ii) (with $k=0$) is clear. Often one can go from this directly to (ii) by means of the law of large numbers. Here, however, the events $\epsilon_n=\{\mathfrak{A}\mid r_n(\mathfrak{A})=0\}$ are not independent and, hence, the law of large numbers does not apply.

The same nonindependence problem arises in (iii) and (iv) and to deal with this problem, one has to derive "quasi-independence" relations. These say in each case that the events are close enough to being independent that one can barely get the desired results without such things as the law of large numbers. §15 of Chapter Three is devoted to quasi-independence. It is essentially new material that has never before been published (and which delayed the completion of the book by several months). The reason for this is that the authors found the original proof of Erdös-Renyi incomplete in that most of the quasi-independence relations were not given and the one that was given was unintelligible. The authors later requested a proof of a special case of that relation and Professor Renyi very kindly supplied it.

Chapter Four covers sieves from small to medium large. It has a beautiful discussion of how present sieves are generalizations of the old sieve of Eratosthenes. The sieves covered are the sieves of Brun and Selberg and the large sieves of Linnik and Renyi. (The book was written before the recent wave of improvements in the large sieve and, indeed, may even have aided it since it was Roth who, after just writing this book, made the first basic improvement in the large sieve.) We will be discussing sieves immediately after the review of this book and so will say no more about them here. It should be noted, however, that Chapter Four itself is quite different from the other chapters both in material and lack of applications. Although quite interesting, it somehow seems out of place in relation to the rest of the book.

Chapter Five covers two related areas. A sequence $\mathfrak{A}$ of positive integers is *primitive* if no element of $\mathfrak{A}$ divides another element of $\mathfrak{A}$. Clearly, a primitive sequence cannot be too thick. The proper measure of this is still another type of density, the *logarithmic density*:

$$\delta_L\mathfrak{A} = \liminf_{n\to\infty} \frac{1}{\log n} \sum_{a_i \leqq n} \frac{1}{a_i}, \qquad \bar{\delta}\mathfrak{A} = \limsup_{n\to\infty} \frac{1}{\log n} \sum_{a_i \leqq n} \frac{1}{a_i},$$

$$\delta\mathfrak{A} = \lim_{n\to\infty} \frac{1}{\log n} \sum_{a_i \leqq n} \frac{1}{a_i},$$

where $\mathfrak{A} = \{a_1, a_2, \cdots\}$. The logarithmic density is more general than the asymptotic density; if $d\mathfrak{A}$ exists, then $\delta\mathfrak{A}$ exists and equals the same thing.

THEOREM. *If $\mathfrak{A}$ is primitive, then $\delta\mathfrak{A} = 0$.*

This might lead one to hope that $d\mathfrak{A} = 0$ but Besicovitch showed $d\mathfrak{A}$ does not always exist.

THEOREM. *If $\mathfrak{A}$ is primitive, then $d_L\mathfrak{A} = 0$, $\bar{d}\mathfrak{A} \leqq \frac{1}{2}$ (a footnote gives a result not proved in the text which implies $\bar{d}\mathfrak{A} < \frac{1}{2}$) but for any $\epsilon > 0$,* *there exist primitive sequences with* $\bar{d}\mathcal{A} > \frac{1}{2} - \epsilon$.

We also have a surprising contrast to the result $\delta\mathcal{A} = 0$ if $\mathcal{A}$ is primitive:

THEOREM (DAVENPORT-ERDÖS). *If* $\bar{\delta}\mathcal{A} > 0$, *then* $\mathcal{A}$ *contains an infinite subsequence* $\{a_{i_j}\}$ *such that* $a_{i_j} \mid a_{i_{j+1}}$ *for all* j.

A closely related topic is the set of multiples of a given sequence. Let $\mathcal{B} = \mathcal{B}(\mathcal{A})$ be the sequence of all positive multiples of elements of $\mathcal{A}$. Let $\mathcal{B}_m$ be the set of multiples of the first m elements of $\mathcal{A}$. Thus $\mathcal{B}_m$ is the union of congruence classes and hence $d\mathcal{B}_m$ exists. Since further, $\mathcal{B}_{m+1} \supseteq \mathcal{B}_m$, we see that

$$b = \lim_{m \to \infty} d\mathcal{B}_m$$

exists. We therefore hope that $d\mathcal{B}$ exists and equals b but as usual, we are disappointed. We do have the

THEOREM. $\delta\mathcal{B}(\mathcal{A}) = d_L\mathcal{B}(\mathcal{A}) = b$ *(and hence if* $d\mathcal{B}(\mathcal{A})$ *exists it equals* b*). If* $\sum_{a_i \in \mathcal{A}} a_i^{-1}$ *converges then* $d\mathcal{B}(\mathcal{A})$ *exists.*

However, we have the result of Besicovitch:

THEOREM. *Given* $\epsilon > 0$, *there exists an infinite sequence* $\mathcal{A}$ *with* $\bar{d}\mathcal{B}(\mathcal{A}) \geqq \frac{1}{2}$ *and* $d_L\mathcal{B}(\mathcal{A}) \leqq \epsilon$.

In fact, the sequence $\mathcal{A}$ that Besicovitch constructs leads to an example of a primitive sequence $\mathcal{A}'$ with $\bar{d}\mathcal{A}' \geqq \frac{1}{2}(1-\epsilon)$ by simply deleting from $\mathcal{A}$ all elements that are multiples of other elements of $\mathcal{A}$. Finally we have the interesting

THEOREM (ERDÖS). *If, for some* c, $A(n) < cn/\log n$ *for* $n > 1$, *then* $d\mathcal{B}(\mathcal{A})$ *exists. However, if* $\psi(n)$ *is a monotonically increasing function of* n *tending to* ∞, *then there exists a sequence* $\mathcal{A}$ *with* $A(n) = O(n\psi(n)/\log n)$ *such that* $d\mathcal{B}(\mathcal{A})$ *does not exist.*

So much for the actual contents of *Sequences*. We have actually only scratched the surface but by now the reader should have an idea of the contents and the wealth of material involved. The book is essentially self-contained and, in fact, the five chapters may be read in any order. Although this is the most "elementary" of the four books reviewed here, nevertheless we still find occasional Fourier series, Parseval's formula, e to complex arguments and gamma functions. And it was also (for me) the hardest of these four books to read. One reason is that many more routine algebraic steps are left to the reader.

But the main problem is notation. Notation comes and notation goes; the same symbol will stand for different things in different places (particularly in different chapters) and the same thing will be written in different ways in different places. These notations are clearly defined each time and, of course, in the different chapters, with different viewpoints it is excusable to change notations. Also, the majority of notations arise in the course of proofs and are kept for only one or two sections and then discarded so that not too much notation must be permanently kept in mind. Still there are occasional proofs which resurrect (with references—sometimes without the page number—rather than definitions) long forgotten notation which force one to reread several pages before proceeding. At this point, it would be nice to be able to look things up in the index but, unfortunately, in the tradition of other Oxford University Press books, there is no index. This is another English tradition that could well be abandoned.

Finally, especially considering the complexity of the notation, the book has surprisingly few errors, be they printing mistakes or others. We find on p. 13 an $\alpha(\alpha-1)$ that should be $\alpha(1-\alpha)$, and on p. 139 (with a similar thing on p. 140) a statement in Theorem 10 that $\sum g_j(x) < \infty$ which seems a strange way to say without explanation that $\sum g_j(x)$ (a series of positive and negative terms) converges. There is also a curious statement on p. 181 that a sum of N positive terms is $\ll N$ with a footnote saying that a previous lemma shows the constant in $\ll$ to be absolute; whereas in actual fact, each term is obviously $\leqq 1$ and the $\ll N$ can be replaced by $\leqq N$. Clearly none of these are really serious and the fact that I had to give these examples serve to show how carefully the authors wrote and proof-read their book.

This text should be a standard reference in this area for years to come; it has compiled an amazing amount of material, most of which is only in the research journals. There is also a large amount of material on sequences not covered (e.g. partitions, arithmetic progressions) for which the authors promised a second volume. Unfortunately (but not surprisingly, considering the work involved), the second volume appears to be even further in the future now than it did when Volume One was completed.

The rest of this review is devoted to sieves and has the main purpose of filling in the gap between 1966 and the present. The idea of sieving occurs first in the ancient "sieve of Eratosthenes." The idea is that to find all the primes p with $\sqrt{x}<p\leqq x$, it is only necessary to "sieve out" from the set of integers $\leqq x$ all multiples of primes

$\leqq \sqrt{x}$. The number of such primes is $(P = \prod_{p \leqq \sqrt{x}} p)$,

$$1 + \pi(x) - \pi(\sqrt{x}) = x \sum_{d|P} \frac{\mu(d)}{d} + \sum_{d|P} \mu(d) \left(\left[\frac{x}{d} \right] - \frac{x}{d} \right)$$

as is easily checked by setting $[x/d] = \sum_{m \leqq x; d|m} 1$ and interchanging the order of summation. The second sum is the "error term" but due to the large (compared to x) number of divisors of P, a decent estimate of the error term is unlikely to be obtained. A similar formula holds if we generalize the problem to sieving the elements $a \leqq x$ of a given sequence $\mathcal{A}$ by primes p from a certain set $\mathcal{P}$ of primes. Namely $(P = \prod_{p \in \mathcal{P}} p)$,

$$\sum_{a \leqq x;\ a \in \mathcal{A};\ (a,\ P)=1} 1 = x \sum_{d|P} \frac{\mu(d)}{f(d)} + \sum_{d|P} \mu(d) R_d(x) \tag{3}$$

where $x/f(d) + R_d(x) = \sum_{a \leqq x; a \in \mathcal{A}; d|a} 1$ and $f(d)$ is a function chosen so that $R_d(x)$ is not "too large." The basic idea of the upper bound sieves of Brun and Selberg is to replace $\mu(d)$ by a function $\lambda(d)$ such that (i) the right side of (3) is increased and (ii) the number of $d|P$ with $\lambda(d) \neq 0$ should be "not too large." These simple modifications lead to beautiful nontrivial results. So far, we have been "sieving out" numbers $a \equiv 0 \pmod{p}$. We might wish to sieve out numbers from other residue classes, say all numbers a in $k(p)$ given residue classes mod p. The sieves of Brun and Selberg give good results only when $k(p)$ is very much smaller than p. The "large sieves" give results for larger values of $k(p)$. The comments in this paragraph are a much condensed version of the introductory section of Chapter Four of *Sequences*, which I again recommend to any reader who wishes to know what sieves are and what they do. As far as the nonlarge sieves go, I recommend the lectures of Selberg [**14**] and Richert [**12**], Richert's paper [**11**] and the forthcoming book on sieves by Halberstam and Richert [**6**].

The large sieve is one of the most popular topics in analytic number theory at the moment. For instance, one can show [**5**] (details to appear in [**6**]) by means of it that every sufficiently large even number is the sum of a prime and a number which is a product of at most three primes; this is the closest approach to Goldbach's conjecture yet. As stated earlier, Davenport's book (with the correction given in *Mathematika* **14** (1967), 229–232) gives the large sieve and Bombieri's application of it with the simplifications up to the middle of 1966 incorporated. Many further simplifications and new results have been obtained since then. I am indebted to H. L. Montgomery for the following references with which I end this review. As to the contents of Davenport's book, we must mention the simplifications by Gallegher in both the sieve [3] and Bombieri's theorem [4]. Also, the average result of Chapter 29 has been changed by Montgomery [10] from an inequality to an asymptotic equality and in the process the use of the large sieve was eliminated. Bombieri and Davenport [2] have achieved formulations of the large sieve which cannot be essentially sharpened. Also, pertinent to Bombieri's theorem are two papers of Montgomery [8], [9] (in the latter, we find the result that for large x, there is a prime between x and $x+x^{3/5+\epsilon}$). More recently, Bombieri [1] has reformulated the large sieve in terms of an improved Bessel's inequality, an idea that had been used earlier by Renyi in more complicated form (see *Sequences*, p. 233, Lemma 6). The large sieve has been extended to algebraic number fields [7], [13], [15] (Wilson follows the paper of Gallagher [4] and generalizes Bombieri's theorem to algebraic number fields).

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