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CLASSIFICATION THEORY FOR HARDY CLASSES OF ANALYTIC FUNCTIONS

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I. **Introduction.** Suppose that W is an open Riemann surface. Denote by $A(W)$ and $M(W)$ the families of single-valued analytic and meromorphic functions on W, respectively. The Hardy class $H_p(W)$, for $0<p<\infty$, is the family of all $f\in A(W)$ for which $|f|^p$ admits a harmonic majorant on W. Let $AB(W)$ be the family of all bounded $f\in A(W)$. Denote by $MB^*(W)$ the family of all $f\in M(W)$ such that $\ln^+|f|$ admits a superharmonic majorant on W. Write $AB^*(W)=A(W)\cap MB^*(W)$. We shall write $W\in O_p$, O_{AB}, O_{AB^*}, O_{MB^*} whenever $H_p(W)$, $AB(W)$, $AB^*(W)$, $MB^*(W)$, respectively, reduces to the constant functions. Finally, as usual, $W\in O_G$ iff W is parabolic.

Now, as is readily verified, $AB(W)\subseteqq H_p(W)\subseteqq H_q(W)\subseteqq AB^*(W)\subseteqq MB^*(W)$ for $0<q<p<\infty$. It follows that

$$O_G \leqq O_{MB^*} \leqq O_{AB^*} \leqq \bigcap_{q>0} O_q \leqq O_p^- \leqq O_p \leqq O_p^+ \leqq \bigcup_{q<\infty} O_q \leqq O_{AB},$$

where $O_p^- = \bigcup\{O_q \mid 0<q<p\}$, $O_p^+ = \bigcap\{O_q \mid p<q<\infty\}$, $0<p<\infty$. It is known that all of these inclusions are strict in the case of arbitrary Riemann surfaces (see Heins [**3**, pp. 34–50] and Sario-Nakai [**7**, pp. 276–280]). The appropriate constructions are Myrberg type surfaces and hence of infinite genus.

If one now restricts W to be of finite genus, the situation changes. First of all, it is now known that $O_G=O_{MB^*}=O_{AB^*}$ (see Sario-Nakai [**7**, p. 280]). Further, Heins [**3**, pp. 50–51] showed next that $O_G<O_1 \leqq O_{AB}$. Aside from these facts, the classification scheme for Hardy classes for Riemann surfaces of finite genus, and thus for plane domains, has remained an open question (see Heins [**3**, p. 50] and Rudin [**6**, p. 49]).

In one of our recent projects, we found a number of results on function-theoretic null-sets and classification theory for H_p classes. In this note we wish to present some of these results. Included will be a partial, though highly suggestive, answer to the open question mentioned

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above. A more detailed development will appear at a later date.

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II. **On removable singularities.** Let S denote the sphere and X the finite plane. Suppose that E is a bounded closed totally disconnected subset of S. We shall write $E \in N_p$ iff $H_p(V) = H_p(V-E)$ for *every* subdomain V of S for which $E \subseteqq V$. $E \in N_B$ and $E \in N_{MB^*}$ are defined similarly using AB and MB^*, respectively. Let $E \in N_G$ iff $\operatorname{Cap}(E) = 0$. Finally, write $E = \mathcal{K}[\lambda_1, \cdots, \lambda_n]$ iff E is concentrated entirely on an n-star formed by n rays emanating from an origin out to ∞ such that the successive angles are $\pi\lambda_1, \cdots, \pi\lambda_n$, $n \geqq 2$, $\lambda_1 + \cdots + \lambda_n = 2$.

The proofs of the following three theorems involve a combination of techniques from classical potential theory and classification theory.

THEOREM 1. *If $E \in N_G$, then $E \in N_{MB^*}$, and conversely.*

THEOREM 2. *The following are equivalent.* (i) $E \in N_p$; (ii) $S - E \in O_p$; *and* (iii) $H_p(V-E) = H_p(V)$ *for some subdomain V of S for which $E \subseteqq V$.*

THEOREM 3. *Suppose that $E_1, \cdots, E_n$ are mutually disjoint and in N_p. Then, $E = \bigcup \{E_k \mid 1 \leqq k \leqq n\} \in N_p$.*

The following theorem is a generalization of a result in Heins [**3**, p. 50].

THEOREM 4. *Let $\Gamma_1, \cdots, \Gamma_n$ be a finite number of disjoint analytic Jordan arcs. Let $E_1, \cdots, E_n$ be bounded closed totally disconnected subsets of S, $E_j \subseteqq \Gamma_j$, $1 \leqq j \leqq n$. Suppose that all $E_j \in N_B$. Then, $E = \bigcup \{E_j \mid 1 \leqq j \leqq n\} \in N_1$.*

This theorem is proved by introduction of appropriate local analytic coordinates near the Γ_k and by use of classical boundary properties of H_p functions as presented, for example, in Golusin [**2**, pp. 345–366] and Privalov [**5**, pp. 53–83].

The next theorem can be effectively used as a fundamental lemma in the study of the H_p classification. Its proof is in part similar to that of Theorem 4 and involves a study of the H_p spaces corresponding to the sectors.

THEOREM 5. *Suppose that $E \in \mathcal{K}[\lambda_1, \cdots, \lambda_n]$ and $E \in N_B$. Let $\lambda = \min \{\lambda_j \mid 1 \leqq j \leqq n\}$ and $1 \leqq p < \infty$. Suppose that $f \in H_p(S-E)$. Then,*

$$f(z) = \sum_{k=0}^{\infty} c_k z^{-k}, \qquad 0 < |z| \leqq \infty,$$

with $c_k=0$ *for all* $k\geqq M_p$, *where* M_1 *is the least integer* $\geqq 1/\lambda$ *and* M_p, $p>1$, *is the least integer* $>1/p\lambda$. *Thus, the complex linear space* $H_p(S-E)$ *is finite-dimensional.*

Of course, Theorem 5 with $n=2$, $\lambda_1=\lambda_2=1$, is essentially Theorem 4. The last two theorems of this section will serve to show that Theorem 3 is in some strong sense best possible. The proofs depend upon constructions that arise in the following section.

THEOREM 6. *Let* $1\leqq p<\infty$. *Suppose that* $E_n\in N_p$ *for all* $n\geqq 1$ *and that the* E_n *are mutually disjoint. Suppose too that* $E=\bigcup\{E_n \mid 1\leqq n<\infty\}$ *is a bounded closed subset of* S. *Then, it can happen that* $E\notin N_q$, *all* $0<q<\infty$.

THEOREM 7. *Suppose that* $E_1\in N_p$ *and* $E_2\in N_p$ *for some* $0<p<\infty$. *It can then happen that* $E_1\cup E_2\notin N_p$ *although* $E_1\cap E_2$ *consists of a single point.*

We can conclude from these theorems that function-theoretic null-sets of type N_p do not have the properties expected by analogy with N_G and N_B.

III. **On the H_p classification.** Our main result on the H_p classification is the following.

THEOREM 8. *For plane domains the following classification scheme holds*:

$$O_G\leqq O_1^- < O_1\leqq O_{3/2}^- < O_{3/2}\leqq O_2^- < O_2\leqq O_{5/2}^- < O_{5/2}$$
$$\leqq O_3^- < O_3 \cdots < \bigcup_{0<p<\infty} O_p < O_{AB}.$$

The most difficult portion of the proof consists of showing that $O_1^- < O_1$. This is handled by use of Riesz α-potentials and the associated potential theory as in Carleson [1, pp. 14–39]. The appropriate construction is a symmetric bounded closed totally disconnected subset E of the real axis such that $E\in N_B$ and $1/z\in H_p(S-E)$ for all $0<p<1$; see Theorem 4. Inequality $O_{k/2}^- < O_{k/2}$ for $k\geqq 3$ is now proved by use of transformation $w=z^{2/k}$.

Because of its rather surprising simplicity, we will now give the proof of inequality $\bigcup_{0<p<\infty} O_p < O_{AB}$. We need the following generalization of a classical theorem of Nevanlinna-Frostman.

LEMMA. *Let* D *be a subdomain of* S, $D\notin O_G$. *Let* K *be a bounded closed subset of* S *with* $\mathrm{Cap}(K)\neq 0$. *Suppose that* $f\in M(D)$ *and* $f[D]\cap K$ *is void. Then,* $f\in MB^*(D)$.

By now, this result is reasonably well known. See, for example, Heins [**4**, pp. 426–428], Rudin [**6**, pp. 48–49], or Sario-Noshiro [**8**, p. 92].

Take any closed totally disconnected set A on the unit circle such that:

(i) the linear measure of A is 0;

(ii) $\mathrm{Cap}(A)\neq 0$;

(iii) $\mathrm{diam}(A)$ is very small;

(iv) A is symmetric relative to 1;

(v) $1\notin A$.

For instance, A can be a Cantor set. Consider the set $F_1\equiv\log(\log A)\subseteqq X$ in the t-plane. The set F_1 has a number of properties: (i) F_1 is bounded away from 0; (ii) $F_1\subseteqq\{t\,|\,\mathrm{Re}(t)>c\}$, some $-\infty<c<0$; (iii) F_1 is concentrated on the lines $\mathrm{Im}(t)=(2n+1)\pi/2$, n integral; (iv) ∞ is a cluster point of F_1. Form the image of F_1 under $z=1/t$ and adjoin $\{0\}$ to get a bounded closed totally disconnected subset E_1 of S. At once, $E_1\in N_B-N_G$ (see Sario-Oikawa [**9**, pp. 289, 291]). Let $W_1=S-E_1$.

For $z\in W_1$, define $f_1(z)=\exp(e^{1/z})$ and $f_2(z)=1/f_1(z)$ omits set A for $z\in W_1$. Hence, by the Lemma, $f_1\in AB^*(W_1)$. Because $MB^*(W_1)$ is a field, $f_2\in AB^*(W_1)$. Let χ_j be the least harmonic majorant of $\ln^+|f_j|$ on W_1 for $j=1$, 2.

Proceed similarly with regard to the set $F_2\equiv\log(i\log A)\subseteqq X$. Here we let $W_2=S-E_2$ and define $f_3(z)=\exp(ie^{1/z})$, $f_4(z)=1/f_3(z)$ for $z\in W_2$. $f_3(z)$ then omits the set A for $z\in W_2$. It follows that f_3 and f_4 are in $AB^*(W_2)$. Let χ_j be the least harmonic majorant of $\ln^+|f_j|$ on W_2 for $j=3$, 4.

Define $W=S-E_1\cup E_2$. At once, $E_1\cup E_2\in N_B$. Since $\sum_{j=1}^4\ln^+|f_j|\leqq\sum_{j=1}^4\chi_j(z)$, $z\in W$, we find that $e^{1/z}\in H_1(W)$. But, then, $e^{1/pz}\in H_p(W)$ for $0<p<\infty$. Hence, $W\in O_{AB}-\bigcup_{0<p<\infty}O_p$.

The proof of Theorem 6 also follows from this construction. In addition, the following weak form of Theorem 8 can be proved by use of just the same simple method and some simple changes of coordinate:

$$O_G\leqq O_{1/2}<O_1<O_2<O_4<O_8<O_{16}<\cdots<\bigcup_{0<p<\infty}O_p<O_{AB}.$$

IV. **Concluding remarks.** Theorem 8 suggests quite strongly that for plane domains $O_p^-<O_p<O_p^+$ when $1\leqq p<\infty$. The case $0<p<1$ remains somewhat mysterious, however. The main reason for this seems to be the distinct lack of nontrivial techniques for the study of H_p classes when $0<p<1$ (see Heins [**3**]).

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