BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 5, September 1971

UNIFORMIZATION IN A PLAYFUL UNIVERSE

BY YIANNIS N. MOSCHOVAKIS[1]

Communicated by Paul J. Cohen, February 26, 1971

It was shown in [1] and [3] that several questions about projective sets can be answered if one assumes the hypothesis of *projective determinacy*. We show here (in outline) that the same hypothesis settles the questions of *uniformization* and *bases* for all analytical classes.

Let $\omega = \{0, 1, 2, \cdots\}$, $R = {}^{\omega}\omega$ (the "reals"), $\mathfrak{X} = X_1 \times \cdots \times X_k$ with $X_i = \omega$ or $X_i = R$ be any product space. We study subsets of these product spaces, i.e. relations of integer and real arguments.

THEOREM 1. *Let n be odd, $n \geqq 1$, assume that every $\mathbf{\Delta}^1_{n-1}$ game is determined. Then for each Π^1_n relation $P \subseteq R \times \mathfrak{X}$, there exists a Π^1_n relation $P^* \subseteq P$ such that*

$$(\exists \alpha) P(\alpha, x) \Leftrightarrow (\exists ! \alpha) P^*(\alpha, x).$$

(For $n = 1$ this is the classical Kondo-Addison Uniformization Theorem, see [8].)

There are many consequences of this result which are well known. The following computation of bases is the corollary which is foundationally most significant.

THEOREM 2. *If every projective game is determined, then every nonempty analytical set has an analytical element.*

More specifically: if n is even, $n \geqq 2$, and every $\mathbf{\Delta}^1_{n-2}$ game is determined, then every nonempty Σ^1_n subset of R contains a Δ^1_n real; if n is odd, $n \geqq 1$, and every $\mathbf{\Delta}^1_{n-1}$ game is determined, then there is a fixed real α_0 such that the singleton $\{\alpha_0\}$ is Π^1_n (so that α_0 is Δ^1_{n+1}) and every nonempty Σ^1_n subset of R contains a real recursive in α_0.

(For $n = 3$, this gives the Martin-Solovay Basis Theorem [5] with Mansfield's improvement [2]. The proofs in these two papers use only the fairly weak hypothesis that there exists a measurable cardinal, or even that for each α, $\alpha^\#$ exists. Our proof depends on the determinacy of a particular $\mathbf{\Delta}^1_2$ game and it can be verified that this game is determined if for every α, $\alpha^\#$ exists.)

AMS 1970 *subject classifications*. Primary 04A15; Secondary 02F35.

Key words and phrases. Analytical relations, uniformization, bases.

[1] The author is a Sloan Foundation Fellow. During the preparation of this paper he was partially supported by NSF Grant #GP-22937.

Our methods combine easily with methods developed by D. A. Martin [4] to yield the following additional result.

THEOREM 3. *Let n be odd, $n \geqq 1$, let δ_n^1 be the supremum of the lengths of the prewellorderings in Δ_n^1, assume that every Δ_{n-1}^1 game is determined. If B_n is the smallest Boolean algebra of sets containing the open sets and closed under $<\delta_n^1$ unions, then $\Delta_n^1 \subseteq B_n$ and each Σ_{n+1}^1 set is the union of δ_n^1 sets in B_n.*

(For $n=3$ this was shown by Martin in [4].)

In the proofs we use the axioms of Zermelo-Fraenkel set theory and the axiom DC of *dependent choices*, but not the full axiom of choice. Thus the results hold in the theory $ZF+DC+$*each game is determined.* For this latter theory, Theorem 3 combines with results of Martin in [4] and ours [6] to give the elegant characterizations (odd n),

$$\Delta_n^1 = B_n,$$

$$P \in \Sigma_{n+1}^1 \Leftrightarrow P = \bigcup_{\xi<\delta_n^1} P_\xi, \quad \text{with each } P_\xi \in B_n.$$ [2]

Full details will appear in [7].

1. **Terminology.** Precise definitions of the classes Σ_n^1, Π_n^1, etc., determinacy, the axiom DC of dependent choices and recursive functions $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ with domain and range any product space can be found in [6].

If $P \subseteq \mathfrak{X}$ is a *pointset* we also think of it as a relation and write interchangeably,

$$x \in P \Leftrightarrow P(x).$$

It will be convenient to use "algebraic" notation for the logical opera-

---

[2] The exact computation of the ordinals δ_n^1 ($n \geqq 1$) poses a very interesting problem. The following facts are known:

(1) In $ZF+DC$, $\delta_1^1 = \aleph_1$ (classical result).

(2) In $ZF+DC+$*Full Determinacy*, $\aleph_1$ and $\aleph_2$ are measurable, hence regular (R. M. Solovay).

(3) In $ZF+DC+$*Full Determinacy*, all δ_n^1 are cardinals, $\delta_n^1 \geqq \aleph_n$ and for odd n, δ_n^1 is regular, [6].

(4) In $ZF+DC$, $\delta_2^1 \leqq \aleph_2$, hence in $ZF+DC+$*Full Determinacy*, $\delta_2^1 = \aleph_2$ (D. A. Martin, unpublished).

(5) In $ZF+DC+$*Full Determinacy*, $\delta_3^1 = \aleph_{\omega+1}$ = the first regular cardinal above $\aleph_2$, [4]!

(6) In $ZF+$*Projective Determinacy*$+$*Full Choice*, $\delta_3^1 \leqq \aleph_3$, [4].

(7) In $ZF+DC+$*Full Determinacy*, for each odd n, $\delta_n^1 = (\lambda_n)^+$ for some cardinal λ_n of cofinality ω (A. S. Kechris, unpublished, using the methods of the present note).

tions on these pointsets, e.g.

$$x \in \exists^{\omega} P \Leftrightarrow (\exists n) P(n, x) \qquad (P \subseteq \omega \times \mathfrak{X}),$$
$$x \in \forall^{\omega} P \Leftrightarrow (\forall n) P(n, x) \qquad (P \subseteq \omega \times \mathfrak{X}),$$
$$x \in \exists^{R} P \Leftrightarrow (\exists \alpha) P(\alpha, x) \qquad (P \subseteq R \times \mathfrak{X}),$$
$$x \in \forall^{R} P \Leftrightarrow (\forall \alpha) P(\alpha, x) \qquad (P \subseteq R \times \mathfrak{X}).$$

Similarly, if Γ is *a class of pointsets*, $\exists^R\Gamma = \{\exists^R P : P \in \Gamma\}$, $\forall^R \exists^R \Gamma = \{\forall^R \exists^R P : P \in \Gamma\}$, etc.

The *dual* class $\breve{\Gamma}$ is defined by $\breve{\Gamma} = \{\mathfrak{X} - P : P \in \Gamma\}$.

A class of pointsets Γ is *adequate* if it contains all recursive pointsets and is closed under conjunction, disjunction, bounded number quantification and substitution of recursive functions. All classes Σ^1_n, Π^1_n, Δ^1_n are adequate, even with $n = 0$ (Σ^1_0 = all *recursively enumerable* sets).

For each Γ, let $\mathbf{\Gamma}$ be the class of all $P \subseteq \mathfrak{X}$ such that for some $Q \subseteq R \times \mathfrak{X}$, $Q \in \Gamma$ and some $\alpha_0 \in R$,

$$P(x) \Leftrightarrow Q(\alpha_0, x).$$

Finally,

$$\mathbf{\Delta} = \{P \subseteq \mathfrak{X} : P \in \mathbf{\Gamma} \text{ and } \mathfrak{X} - P \in \mathbf{\Gamma}\}.$$

2. **Norms and scales.** The idea of the proof is to formulate a strong *prewellordering property*, like that of [**1**], which on the one hand can be shown to propagate from each Π^1_n to Σ^1_{n+1} and from each Σ^1_{n+1} to Π^1_{n+2}, and on the other hand implies uniformization when it holds on a Π-class.

A *norm* on a set P is a function $\varphi : P \rightarrow$ordinals; we call φ a Γ-*norm* if there are relations $\leqq_\Gamma$, $\leqq_{\breve{\Gamma}}$ in Γ and $\breve{\Gamma}$ respectiveley, such that

$$P(y) \Rightarrow (\forall x)[x \leqq_\Gamma y \Leftrightarrow x \leqq_{\breve{\Gamma}} y \Leftrightarrow [P(x) \;\&\; \varphi(x) \leqq \varphi(y)]].$$

Γ has the *prewellordering property* in the sense of [**1**] or [**3**], if every $P \in \Gamma$ admits a Γ-norm.

A scale on a set P is a sequence $\varphi_0, \varphi_1, \varphi_2, \cdots$ of norms on P such that the following *limit condition* holds:

(*) If $x_0, x_1, x_2, \cdots \in P$, if $\text{limit}_{i \to \infty} x_i = x$, if, for each n and all large i, $\varphi_n(x_i) = \lambda_n$, then $P(x)$ and, for each n, $\varphi_n(x) \leqq \lambda_n$.[3]

We call $\varphi_0, \varphi_1, \varphi_2, \cdots$ a Γ-*scale* if there are relations $S_\Gamma(n, x, y)$, $S_{\breve{\Gamma}}(n, x, y)$ in Γ and $\breve{\Gamma}$ respectively, such that for each n,

[3] I wish to thank my student A. S. Kechris for simplifying my original definition of a scale and thereby shortening considerably the computation in the proof of C below.

$$P(y) \Rightarrow (\forall x)[S_\Gamma(n, x, y) \Leftrightarrow S_{\breve{\Gamma}}(n, x, y) \Leftrightarrow [P(x) \,\&\, \varphi_n(x) \leqq \varphi_n(y)]].$$

Γ has *property* $\mathcal{S}$ if each $P \in \Gamma$ admits a Γ-scale.

3. **Basic results.** Theorem 1 follows fairly easily from the following four basic results.

THEOREM A. *The class Σ_0^1 of all recursively enumerable sets has property* $\mathcal{S}$.

THEOREM B. *If Γ is adequate, $P \in \Gamma$ and P admits a Γ-scale, then $\exists^R P$ admits a $\exists^R \forall^R \Gamma$-scale.*

THEOREM C. *If Γ is adequate, if each Δ game is determined and DC holds, if $P \in \Gamma$ admits a Γ-scale, then $\forall^R P$ admits a $\forall^R \exists^R \Gamma$-scale.*

THEOREM D. *If Γ is adequate, $\exists^\omega \Gamma \subseteq \Gamma$, $\forall^\omega \Gamma \subseteq \Gamma$, $\forall^R \Gamma \subseteq \Gamma$ and Γ has property $\mathcal{S}$, then for each $P \subseteq R \times \mathfrak{X}$, $P \in \Gamma$, there is some $P^* \subseteq P$ such that $P^* \in \Gamma$ and*

$$(\exists \alpha) P(\alpha, x) \Leftrightarrow (\exists ! \alpha) P(\alpha, x).$$

4. **Proofs.** Proof of A is trivial and that of D is a minor modification of a standard proof of the Kondo-Addison Theorem, e.g. that in [8]. Proofs of B and C are elaborations of the corresponding cases in the proof of the Prewellordering Theorem in [1]. We omit all details of B, which is the easier of the two.

To prove C, suppose

$$P(x) \Leftrightarrow (\forall \alpha) Q(\alpha, x),$$

with $Q \in \Gamma$, let $\psi_0, \psi_1, \psi_2, \cdots$ be a Γ-scale on Q. Let $u_0, u_1, u_2, \cdots$ be a recursive enumeration of all finite sequences of ω such that u_0 is the empty sequence and if u_i is an initial segment of u_j, then $i < j$. For each i and each x, y, consider the game $G_i(x, y)$ defined as follows: if player I plays γ and player II plays δ, put

$$\alpha = u_i {}^\frown \gamma, \qquad \beta = u_i {}^\frown \delta$$

and call II a winner if one of the following conditions hold:

(0) $\neg Q(\beta, y)$,

(1) $Q(\beta, y) \,\&\, Q(\alpha, x) \,\&\, \psi_0(\alpha, x) < \psi_0(\beta, y)$,

(2) $Q(\beta, y) \,\&\, Q(\alpha, x) \,\&\, \psi_0(\alpha, x) = \psi_0(\beta, y) \,\&\, \psi_1(\alpha, x) < \psi_1(\beta, y)$,

$\cdots \qquad \cdots$

(i) $$Q(\beta, y) \,\&\, Q(\alpha, x) \,\&\, \psi_0(\alpha, x) = \psi_0(\beta, y) \,\&\, \cdots \,\&\, \psi_{i-1}(\alpha, x) = \psi_{i-1}(\beta, y) \,\&\, \psi_i(\alpha, x) \leqq \psi_i(\beta, y).$$

For each i, put

$$P_i(x) \Leftrightarrow (\forall \alpha \supseteq u_i) Q(\alpha, x),$$

$$x \leqq_i y \Leftrightarrow x, y \in P_i \ \& \ \text{II wins } G_i(x, y).$$

Notice that $P_0 = P$. Now the methods of [1] easily show that if every Δ game is determined, then each $\leqq_i$ is a prewellordering on P_i and hence defines a norm $\varphi_i : P_i \to$ordinals. Moreover, there are relations $S_1(n, x, y)$, $S_2(n, x, y)$ in $\forall^R \exists^R \Gamma$ and $\exists^R \forall^R \breve{\Gamma}$ respectively, such that

$$P_n(y) \Rightarrow (\forall x)[S_1(n, x, y) \Leftrightarrow S_2(n, x, y) \Leftrightarrow [P_n(x) \ \& \ \varphi_n(x) \leqq \varphi_n(y)]].$$

The sequence $\varphi_0, \varphi_1, \varphi_2, \cdots$ consists of norms on different sets, but it is not hard to verify that if we can show the limit property (*) for it, then we can define a scale $\varphi_0', \varphi_1', \varphi_2', \cdots$ on P itself.

Let $x_0, x_1, x_2, \cdots \in P$, assume that $\text{limit}_{i\to\infty} x_i = x$ and for each n and all large i, $\varphi_n(x_i) = \lambda_n$; we must show that $P(x)$ and for all n, $\varphi_n(x) \leqq \lambda_n$. Without loss of generality we may assume that $\varphi_n(x_i) = \lambda_n$, all $i \geqq n$; thus it is enough to show that, for each i, II has a winning strategy in $G_i(x, x_i)$, since for $i=0$ this proves $P(x)$ and for all i it shows $x \leqq_i x_i$, i.e. $\varphi_i(x) \leqq \varphi_i(x_i) = \lambda_i$.

Suppose $u_i = (a_0, \cdots, a_l)$ and let us picture the game $G_i(x, x_i)$ as follows:

$$G_i(x, x_i) \quad \begin{cases} a_0, a_1, \cdots, a_l & \mathrm{I}(x) \quad a_{l+1}, a_{l+2}, a_{l+3}, \cdots, & \alpha \\ a_0, a_1, \cdots, a_l & \mathrm{II}(x_i) \quad \alpha_1(l+1), \alpha_1(l+2), \alpha_1(l+3), \cdots, & \alpha_1. \end{cases}$$

Here I's first move is labeled a_{l+1}, his second a_{l+2}, etc. Let $j_1, j_2, \cdots$ be chosen so that

$$u_{j_{n+1}} = (a_0, a_1, \cdots, a_l, a_{l+1}, \cdots, a_{l+n});$$

notice that $i = j_1 < j_2 < j_3 < \cdots$ and that j_{n+1} is known as soon as a_{l+n} has been played. For each n then, II simulates on the side the game $G_{j_n}(x_{j_{n+1}}, x_{j_n})$ in which the second player has a winning strategy. In all these simulated games, the second player follows some winning strategy. The first player starts with a_{l+n} and then continues by copying the second player's moves in $G_{j_{n+1}}(x_{j_{n+2}}, x_{j_{n+1}})$ as in the diagram below. Finally II copies the second player's move in $G_{j_1}(x_{j_2}, x_{j_1})$ for the original game $G_i(x, x_i)$.

$$\begin{array}{lllll}
G_{j_1}(x_{j_2}, x_{j_1}) & \begin{cases} a_0, \cdots, a_l \\ a_0, \cdots, a_l \end{cases} & \begin{array}{l} \mathrm{I}(x_{j_2}) \\ \mathrm{II}(x_{j_1} = x_i) \end{array} & \begin{array}{l} a_{l+1}, \alpha_2(l+2), \alpha_2(l+3), \cdots, \\ \alpha_1(l+1), \alpha_1(l+2), \alpha_1(l+3), \cdots, \end{array} & \begin{array}{l} \alpha_2 \\ \alpha_1 \end{array} \\
G_{j_2}(x_{j_3}, x_{j_2}) & \begin{cases} a_0, \cdots, a_l, a_{l+1} \\ a_0, \cdots, a_l, a_{l+1} \end{cases} & \begin{array}{l} \mathrm{I}(x_{j_3}) \\ \mathrm{II}(x_{j_2}) \end{array} & \begin{array}{l} a_{l+2}, \alpha_3(l+3), \cdots, \\ \alpha_2(l+2), \alpha_2(l+3), \cdots, \end{array} & \begin{array}{l} \alpha_3 \\ \alpha_2 \end{array} \\
G_{j_3}(x_{j_4}, x_{j_3}) & \begin{cases} a_0, \cdots, a_l, a_{l+1}, a_{l+2} \\ a_0, \cdots, a_l, a_{l+1}, a_{l+2} \end{cases} & \begin{array}{l} \mathrm{I}(x_{j_4}) \\ \mathrm{II}(x_{j_3}) \end{array} & \begin{array}{l} a_{l+3}, \cdots, \\ \alpha_3(l+3), \cdots, \end{array} & \begin{array}{l} \alpha_4 \\ \alpha_3 \end{array}
\end{array}$$

At the end the second players have won all the simulated games and reals $\alpha, \alpha_1, \alpha_2, \alpha_3, \cdots$ have been defined. Clearly $\text{limit}_{i\to\infty} \alpha_i = \alpha$, so that $\text{limit}_{i\to\infty} (\alpha_i, x_i) = (\alpha, x)$. It is now easy to verify that all norms $\psi_n(\alpha_i, x_i)$ are constant for all large i, so that $Q(\alpha, x)$, and furthermore that II wins $G_i(x, x_i)$, thus completing the proof.

(ADDED IN PROOF, June 27, 1971.) K. Kunen and D. A. Martin have now shown, independently, in *ZF+DC+Projective Determinacy*, that for each odd n, $\boldsymbol{\delta}_{n+1} \leqq (\delta_n)^+$; their proofs use the methods of this note. By entirely different methods D. A. Martin also showed in *ZF+DC+Full Determinacy*, that for each odd n, $\boldsymbol{\delta}_n^1$ is measurable, and K. Kunen showed that under the same hypotheses for all $n \geqq 1$, $\boldsymbol{\delta}_n^1$ is measurable.

BIBLIOGRAPHY

1. J. W. Addison and Yiannis N. Moschovakis, *Some consequences of the axiom of definable determinateness*, Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 708–712. MR **36** #4979.

2. Richard Mansfield, *A Souslin operation for* Π_2^1, Israel J. Math. **9** (1971), 367–379.

3. D. A. Martin, *The axiom of determinateness and reduction principles in the analytical hierarchy*, Bull. Amer. Math. Soc. **74** (1968), 687–689. MR **37** #2607.

4. ———, *Pleasant and unpleasant consequences of determinateness*, unpublished manuscript, circulated in March 1970.

5. D. A. Martin and R. M. Solovay, *A basis theorem for* Σ_3^1 *sets of reals*, Ann. of Math. (2) **89** (1969), 138–159. MR **41** #53.

6. Yiannis N. Moschovakis, *Determinacy and prewellorderings of the continuum*, Math. Logic and Foundations of Set Theory, North-Holland, Amsterdam and London, 1970, pp. 24–62.

7. ———, *Descriptive set theory, a foundational approach*, Studies in Logic, North-Holland, Amsterdam (to appear).

8. Joseph R. Shoenfield, *Mathematical logic*, Addison-Wesley, Reading, Mass., 1967. MR **37** #1224.

UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024