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CONCORDANCE-HOMOTOPY GROUPS AND THE NONFINITE TYPE OF SOME $\mathrm{Diff}_0 M^n$

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1. **Introduction.** Let M^n be a closed, oriented, C^∞ n-manifold, and let X be a compact C^∞ submanifold of M^n of codimension zero. $\mathrm{Diff}(M^n, X)$ is the group of diffeomorphisms $M^n \to M^n$ that preserve orientation and fix each point of X. In [**1**] and [**2**], we show that the homotopy groups $\pi_i(\mathrm{Diff}(M^n, X))$ are nontrivial for certain i, M^n and X. From this we deduce that $\mathrm{Diff}_0(M^n, X)$, the identity component of $\mathrm{Diff}(M^n, X)$, has nonfinite homotopy type in certain cases. In this announcement, we describe machinery, of interest in its own right, that allows us to sharpen and extend some of the results of [**2**]. A detailed exposition will appear elsewhere.

2. **Concordance-homotopy groups of geometric automorphism groups.** Say that a self-map $f: Z \to Z$ *has compact support* if it coincides with the identity outside some compact subset of Z. Homotopy classes in $\pi_i(\mathrm{Diff}(M^n, X))$, $i \geqq 0$, may be represented by compactly supported, C^∞ bundle equivalences $(M^n - X) \times \boldsymbol{R}^i \to (M^n - X) \times \boldsymbol{R}^i$ over the Euclidean space $\boldsymbol{R}^i$. Homotopies may be described similarly as certain bundle equivalences over $\boldsymbol{R}^i \times [0, 1]$. If we weaken these notions by not requiring that the $\boldsymbol{R}^i$ (or $\boldsymbol{R}^i \times [0, 1]$) parameters be preserved, then we obtain a "concordance-homotopy group" which we denote by $\pi_i(\mathfrak{Diff}; M^n \text{ rel } X)$. This group is also described in [**2**]. Here, $\mathfrak{Diff}$ denotes the category of oriented, C^∞ manifolds and orientation-preserving diffeomorphisms.

We may extend this definition directly to the categories $\mathcal{PL}$ (=PL manifolds and PL isomorphisms), $\mathfrak{Top}$ (= topological manifolds and homeomorphisms), and $\mathcal{H}$ (=topological manifolds and homotopy equivalences), in which all manifolds are oriented and all maps preserve orientation. Similarly, using Euclidean half-spaces, we may de-

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fine "relative" groups $\pi_{i+1}(\mathcal{B}, \mathcal{A}; M^n \text{ rel } X)$, $i \geqq 0$, where $(\mathcal{B}, \mathcal{A})$ is one of the pairs of categories $(\mathcal{PL}, \mathfrak{Diff})$, $(\mathfrak{Top}, \mathfrak{Diff})$, $(\mathcal{H}, \mathfrak{Diff})$, $(\mathfrak{Top}, \mathcal{PL})$, $(\mathcal{H}, \mathcal{PL})$ or $(\mathcal{H}, \mathfrak{Top})$. Here, of course, M^n is assumed to be a closed, oriented $\mathcal{A}$-manifold and X is a locally-flat, codimension-0, $\mathcal{A}$-submanifold of M^n.

The following theorem generalizes a result of Hodgson [5], [6], who considers these groups from a slightly different point of view.

2.1 THE EXACTNESS THEOREM. *Let* $(\mathcal{B}, \mathcal{A})$, M^n *and* X *be as above. Then, there is a long exact sequence of groups and homomorphisms*

$$\cdots \xrightarrow{j} \pi_{i+1}(\mathcal{B}, \mathcal{A}; M \text{ rel } X) \xrightarrow{\partial} \pi_i(\mathcal{A}; M \text{ rel } X) \xrightarrow{k} \pi_i(\mathcal{B}; M \text{ rel } X) \xrightarrow{j} \cdots.$$

2.2 REMARKS. (a) When $i \geqq 1$, the groups in the above sequence, which we call *the* $(\mathcal{B}, \mathcal{A})$*-sequence for* (M^n, X), are abelian.

(b) When $(\mathcal{B}, \mathcal{A}) \neq (\mathcal{PL}, \mathfrak{Diff})$, the homomorphisms in the sequence are defined, essentially, by inclusions or restrictions. When $(\mathcal{B}, \mathcal{A}) = (\mathcal{PL}, \mathfrak{Diff})$, some nontrivial applications of the theory of C^1-triangulations [12] are needed to define the maps (and the relative groups). In this case we assume that the C^∞ manifold M^n has been endowed with a PL structure via a smooth triangulation.

(c) The $(\mathcal{B}, \mathcal{A})$ sequences are natural with respect to weakening of the pair $(\mathcal{B}, \mathcal{A})$ and suitable maps of pairs $(M_1^n, X_1) \to (M_2^n, X_2)$.

We shall assume the standard notation of homotopy theory. Corresponding to each category $\mathcal{A} = \mathfrak{Diff}$, $\mathcal{PL}$, $\mathfrak{Top}$, or $\mathcal{H}$, there is a "stable group" $A = O$, PL, Top, G. We use these in the statement of our next result:

2.3 THE CLASSIFICATION THEOREM. *Let* $(\mathcal{B}, \mathcal{A})$, M^n *and* X *be as above, and, for simplicity, suppose that* $M^n - X$ *is connected. For every* $i \geqq 1$, *there exist homomorphisms*

$$\kappa_i : \pi_i(\mathcal{B}, \mathcal{A}; M \text{ rel } X) \to [\Sigma^i(M/X), B/A]$$

satisfying:

(a) *If* $n+i \geqq 6$ *and* $\mathcal{B} \neq \mathcal{H}$, *then* κ_i *is an isomorphism.*

(b) *If* $n+i \geqq 6$ *and* $\mathcal{B} = \mathcal{H}$, *then there is an exact sequence of groups and homomorphisms*

$$\begin{array}{ccccc} [\Sigma^{i+1}(M/X), G/A] & \xrightarrow{\theta} & L^s_{n+i+1}(\pi) & \to & \pi_i(\mathcal{H}, \mathcal{A}; M \text{ rel } X) \\ & & & & \downarrow \kappa_i \\ & & & & [\Sigma^i(M/X), G/A] \xrightarrow{\theta} L^s_{n+i}(\pi), \end{array}$$

where θ *is the surgery obstruction homomorphism* (*cf.* [11, *p.* 10.15] *and*

$L_k^s(\pi)$ is the kth Wall group (for simple homotopy equivalence) associated to $\pi=\pi_1(M^n-X)$.

2.4 REMARKS. (a) κ_i is natural with respect to weakening of the categories and suitable maps of pairs.

(b) The theorem is proved by constructing a natural *bijection* between $\pi_i(\mathcal{B}, \mathcal{A}; M \text{ rel } X)$ and the "structure groups"

$$\mathcal{S}(\mathcal{B}, \mathcal{A}; N\times D^i, \partial(N\times D^i))$$

(see [8], [10], [11]), where $N=M^n-\text{int } X$, and then applying results of [8], [9], [10], [11]. Without the dimension restriction $n+i \geqq 6$, we can only obtain an injection: the proof of surjectivity uses the s-cobordism theorem.

(c) Sullivan [10] obtains 2.3(b) when $(\pi, i)=(0, 1)$ (cf. [6]).

3. **Some computations.** Let $D^n_\pm$ be complementary hemispheres in S^n.

3.1 PROPOSITION.

$$\pi_i(\mathcal{A}; S^n \text{ rel } D^n_+) = 0, \qquad \mathcal{A} \neq \mathfrak{Diff},$$
$$= \Gamma^{n+i+1}, \quad \mathcal{A} = \mathfrak{Diff}.$$

PROOF. The case $\mathcal{A}=\mathfrak{Diff}$ is mentioned in [2]. The other cases are proved using the Alexander trick. Here, as in [2], Γ^k is the Kervaire-Milnor-Smale group.

If $(\mathcal{B}, \mathcal{A})$ is a pair of categories as in §2, then 2.1 and 3.1 imply that $\pi_{i+1}(\mathcal{B}, \mathcal{A}; S^n \text{ rel } D^n_+)\approx\pi_i(\mathcal{A}; S^n \text{ rel } D^n_+)$.

3.2 PROPOSITION. *If M^n is a $K(\pi, 1)$, then $\pi_i(\mathcal{H}; M^n)=0$, $i>1$.*

PROOF. Use obstruction theory.

3.3 PROPOSITION. *If M^n is a* PL *homotopy n-torus, then*

(a) $$\pi_i(\mathcal{H}, \mathcal{PL}; M^n) \approx H^{3-i}(M^n; Z_2), \qquad n+i \geqq 6,$$

(b) $$\pi_i(\mathcal{PL}; M^n) \approx 0, \quad i>2; \qquad \approx Z_2, \quad i=2, \qquad n+i\geqq 6.$$

PROOF. Part (b) follows immediately from (a), 3.2, and the Exactness Theorem. To prove (a), we may refer to the sequence in 2.3(b), and calculate the desired result exactly as in [7], or we simply combine remark 2.4(b) with the results of [7]: these results give (a) with π_i replaced by $\mathcal{S}$. Q.E.D.

Now let $e: D^n_- \to M^n-X$ be an orientation-preserving C^∞ embedding and define $p:(M, X)\to(S^n, D^n_+)$ to be the composite

$$(M^n, X) \subseteq (M^n, M^n - \operatorname{int} e(D^n_-)) \xrightarrow{\pi} (S^n, D^n_+),$$

where π is any map of pairs coinciding with e^{-1} on image e. Clearly p has degree one.

Then, essentially by conjugating with p, we obtain a homomorphism $p^\#$ from the $(\mathcal{B}, \mathfrak{Diff})$-sequence for (S^n, D^n_+) to the $(\mathcal{B}, \mathfrak{Diff})$-sequence for (M, X). On $\pi_i(\mathfrak{Diff};\ S^n \text{ rel } D^n_+) \approx \Gamma^{n+i+1}$, $p^\#$ is precisely the homomorphism $\mathcal{E}_*$ of [2]. Let $p': M/X \to S^n$ be the degree-one map determined by p; right-composition with $\Sigma^k p'$ determines maps

$$p^*: \pi_{n+k}(Y) \to [\Sigma^k(M/X), Y],$$

for every pointed space Y. Now consider $p^\#$ on $\pi_{i+1}(\mathcal{B}, \mathfrak{Diff};\ S^n \text{ rel } D^n_+)$.

3.4 Proposition. *Suppose that the tangent bundle of $M^n - X$ is s.f.h.t. (= stably fibre-homotopy trivial) and that $n+i \geqq 5$. Then*:

(a) *When $\mathcal{B} \neq \mathcal{H}$, $p^\#$ is a split-injection*;

(b) *When $\mathcal{B} = \mathcal{H}$, $\ker p^\# \subseteq bP_{n+i+2} \subseteq \Gamma^{n+i+1} = \pi_{i+1}(\mathcal{B}, \mathfrak{Diff};\ S^n \text{ rel } D^n_+)$.*

This result follows easily from 2.3(a) and (b) together with the following lemma.

3.5 Lemma. *If the tangent bundle of $M^n - X$ is s.f.h.t., then the homomorphism*

$$p^*: \pi_{n+k}(B/O) \to [\Sigma^k(M^n/X), B/O]$$

is a split-injection.

Proof. The tangent bundle of $M^n - X$ is s.f.h.t. if and only if the top-dimensional homology generator of $\Sigma^K(M/X)$, k sufficiently large, is spherical, which is true if and only if $\Sigma^K p'$ has a right-inverse. The desired result now follows by observing that B/O is an infinite loop space [3]. Q.E.D.

4. Applications to some $\pi_i(\mathrm{Diff}(M^n, X))$. Let M^n and X be as in the introduction, and suppose always that $n+i \geqq 5$. Let $\mathcal{B}$ be any one of the categories $\mathcal{PL}$, $\mathfrak{Top}$, or $\mathcal{H}$. We have a commutative diagram

$$\begin{array}{ccccc}
\pi_i(\mathrm{Diff}(S^n, D^n_+)) & \xrightarrow{\Phi} & \pi_i(\mathfrak{Diff};\ S^n \text{ rel } D^n_+) & \xleftarrow{\partial} & \pi_{i+1}(\mathcal{B}, \mathfrak{Diff};\ S^n \text{ rel } D^n_+) \\
E_* \downarrow & & \downarrow p_0^\# & & \downarrow p^\# \\
\pi_i(\mathrm{Diff}(M^n, X)) & \xrightarrow{\Phi} & \pi_i(\mathfrak{Diff};\ M^n \text{ rel } X) & \xleftarrow{\partial} & \pi_{i+1}(\mathcal{B}, \mathfrak{Diff};\ M \text{ rel } X) \\
 & & & & \uparrow k \\
 & & & & \pi_{i+1}(\mathcal{B};\ M \text{ rel } X)
\end{array}$$

The left-hand square is the diagram in §3 of [2].

4.1 LEMMA. *Let M^n be a $K(\pi, 1)$ with s.f.h.t. tangent bundle (cf. 3.4). Then, for $i \geqq 1$,*

$$\text{kernel } p_0^{\#} \subset bP_{n+i+2} \subset \Gamma^{n+i+1} = \pi_i(\mathfrak{Diff}; S^n \text{ rel } D_+^n).$$

PROOF. Set $\mathcal{B} = \mathcal{H}$ in the above diagram. The result then follows immediately from 3.1, 3.2, and 3.4(b). Q.E.D.

4.2 LEMMA. *Let M^n be a smooth, homotopy n-torus. Then:*

(a) *If $i \geqq 2$,* kernel $p_0^{\#} = 0$;

(b) *If $i = 1$,* kernel $p_0^{\#} = 0$ *or* $\boldsymbol{Z}_2$.

PROOF. This follows immediately from the diagram by letting $\mathcal{B} = \mathcal{PL}$ and applying 3.3(b) and 3.4(a). Q.E.D.

The subgroup $\Phi(\pi_i(\text{Diff}(S^n, D_+^n))) \subseteq \Gamma^{n+i+1}$ has been denoted by Γ_{i+1}^{n+i+1} by Gromoll [4]. In [1, 1.1 and 1.2], we show that many of these Gromoll subgroups are nonzero and, in fact, that some of them are not contained in bP_{n+i+2} [1, 1.2]. These facts together with 4.1 and 4.2 yield nontriviality results for certain $\pi_i(\text{Diff}(M^n, X))$. Indeed, one can also deduce that some of these groups have nontrivial torsion (cf. [1, 2.2]). We shall not list all of these results but only the following sample corollaries:

4.3 COROLLARY. *Let M^n be a smooth homotopy n-torus.*

(a) *If $n \geqq 7$, then M^n and* $\text{Diff}_0 M^n$ *have distinct homotopy types.*

(b) *If $n = 4k$, $k \geqq 3$ or $n = 8k - 6$, k not a power of two, then* $\text{Diff}_0 M^n$ *does not have finite type.*

This strengthens a special case of Theorem 2.1 of [2] (cf. Remark 2.3 of [2]).

4.4 COROLLARY. *Let T^n be the standard, C^∞ n-torus, and let $T_+ = T^{n-1} \times D_+^1 \subset T^n$. Then:*

(a) $\text{Diff}(T^n, T_+)$ *is homotopy abelian;*

(b) $\text{Diff}_0(T^n, T_+)$ *does not have finite type when $n \geqq 7$.*

Let us call the pair of natural numbers (n, i) allowable if there exist an odd prime Q and integers u and v satisfying the following (cf. [1, 1.2]):

(i) $0 \leqq v < u \leqq Q - 1$, $u - v \neq Q - 1$,

(ii) $i \leqq 2Q - 3$,

(iii) $n + i = 2(uQ + v + 1)(Q - 1) - 2(u - v) - 2$.

4.5 COROLLARY. *Let M^n be a $K(\pi, 1)$ with s.h.f.t. tangent bundle (see Proposition 3.4). If (n, i) is allowable, then $\pi_i(\text{Diff } M^n)$ contains odd torsion. If $(n, 2)$ is allowable, then* $\text{Diff}_0 M^n$ *does not have finite type.*

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