BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 2, March 1971

THE CONJECTURES ON CONFORMAL TRANSFORMATIONS OF RIEMANNIAN MANIFOLDS

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Communicated by Shlomo Sternberg, July 23, 1970

0. **Introduction.** Let (M, g) be a Riemannian manifold. If a Riemannian metric g^* on M is related to g as $g^*=e^{2\rho}g$ for some smooth function ρ on M, then g^* is said to be *conformal* to g. This relation is obviously an equivalence relation in the set of all the Riemannian metrics on M.

Let h be a C^∞-map of (M, g) into another Riemannian manifold (M', g'). If the Riemannian metric h^*g' induced on M by h is conformal to g, h is called a *conformal map* of (M, g) into (M', g'). Obviously h is conformal if and only if it preserves the angle between any two tangent vectors. h remains conformal under any conformal changes of metrics on M and M'. If h is a diffeomorphism, then it is called a *conformal diffeomorphism*, or briefly a *conformorphism* of (M, g) onto (M', g'), and (M, g) is said to be *conformally diffeomorphic*, or *conformorphic* to (M', g') through h. If $(M, g)=(M', g')$, then h is called a *conformal transformation* or a *conformorphism* of (M, g).

It is known that the group $C(M, g)$ of all conformorphisms of (M, g) is a Lie group with respect to the compact-open topology. $C_0(M, g)$ denotes its connected component of the identity. Let G be a subgroup of $C(M, g)$. G is said to be *essential* if for any smooth function ρ, G is not contained in $I(M, e^{2\rho}g)$, the group isometries of $(M, e^{2\rho}g)$. Otherwise it is inessential.

There have been two famous conjectures.

CONJECTURE I. *Let (M, g) be a connected compact Riemannian n-manifold, $n>2$. Then $C_0(M, g)$ is essential if and only if (M, g) is conformorphic to a Euclidean n-sphere.*

CONJECTURE II. *Let (M, g) be a connected compact Riemannian n-manifold, $n>2$, with constant scalar curvature k. Then $C_0(M, g)$ is essential if and only if k is positive and (M, g) is isometric to a Euclidean n-sphere $S^n(k)$ of radius $1/k^{1/2}$.*

The purpose of this paper is to prove these conjectures affirmatively.

As for Conjecture I, Nagano [6] and Ba [2] proved it in the case of homogeneous Riemannian manifolds. Avez [1] and Obata [9] gave

AMS 1970 *subject classifications*. Primary 5372, 5325; Secondary 5747.
Key words and phrases. Conformorphism, essential group of conformorphisms.

an answer under the condition that the vector field defined by a one-parameter group of conformorphisms has nonvanishing divergence at each of its singular points.

We cite a result for later use.

THEOREM 0 [10]. *Let M be a conformally flat Riemannian n-manifold, $n>2$, with a finite fundamental group. If M admits an essential one-parameter group of conformorphisms, then M is conformorphic to S^n or $S^n - \{p_\infty\}$.*

The assumption of a finite fundamental group will be removed later.

Ledger-Obata [3] gave an affirmative answer under additional assumptions of analyticity and a finite fundamental group of the manifold under consideration. Lelong-Ferrand [4] announced, without detailed proof, that this is true. Our proof given below is different from Lelong-Ferrand's and is essentially the same as Ledger and Obata's with no further assumptions.

As for Conjecture II, many approaches have been made. In most cases, affirmative answers were given under additional conditions on the Riemannian metric in question such as being of Einstein, with parallel Ricci tensor, homogeneous Riemannian, with constant length of the curvature tensor or Ricci tensor, etc. In many cases, the problem is reduced to the existence of a gradient conformal vector field. It is remarked that a one-parameter group generated by a gradient conformal vector field is essential and closed in $C_0(M, g)$.

Our proof given below is different from the above. Once Conjecture I is established, Conjecture II is true if the following is true.

CONJECTURE II'. *Let (S^n, g) be a Euclidean n-sphere, $n>2$, of radius 1. If g^* is another Riemannian metric on S^n conformal to g, then g^* is of constant scalar curvature 1 if and only if it is of constant curvature 1.*

Furthermore if this is the case g^* is obtained by an element σ of $C(S^n, g)$ from g, i.e. $g^* = \sigma^* g$.

In this paper we will give affirmative answers to the above conjectures with outlines of the proofs. The details will be given elsewhere.

1. Conjecture I.

THEOREM I. *Conjecture* I *is true.*

The proof is divided into several steps.

PROPOSITION 1.1. *Let (M, g) be as in Conjecture* I. *If $C_0(M, g)$ is essential, then (M, g) is conformally flat.*

To prove this, let W be the Weyl's conformal curvature tensor of type (1, 3), whose vanishing implies conformally-flatness for $n>3$. In case $n=3$ it vanishes identically and is replaced by a tensor of type (0, 3), denoted by the same letter, whose vanishing gives conformally-flatness as well. By this understanding, we put $M_1=\{p\in M: W_p\neq 0\}$. To prove the proposition, we show that M_1 is empty.

Suppose that M_1 is not empty. Let (M_1, g_1) denote the Riemannian open submanifold of (M, g), where g_1 is the restriction of g to M_1. Since W is invariant by any conformorphism, M_1 is invariant by $C(M, g)$. Therefore there is a map $\alpha: C(M, g)\rightarrow C(M_1, g_1)$.

LEMMA 1.2. *α is a closed map.*

The proof is based on a fact that the conformal structure is a G-structure of finite type, though it is omitted here.

Once Lemma 1.2 is established, the proof of Proposition 1.1 goes through in the same way as in the corresponding part of Ledger-Obata [**3**]. In fact in [**3**], to show that α is closed, the assumptions that M is analytic and has a finite fundamental group are used. The proof of Proposition 1.1 goes as follows. Since $C_0(M, g)$ is not compact, there exists a closed one-parameter subgroup G of $C_0(M, g)$ which is isomorphic to the additive group R of the real numbers. Then by Lemma 1.2, G is considered as a closed subgroup of $C_0(M_1, g_1)$. Since W never vanishes on M_1, $C(M_1, g_1)=I(M_1, g_1')$ where $g_1'=\|W\|g_1$, $\|W\|$ being the length of W. Therefore G is a closed subgroup of $I(M_1, g_1')$ and hence every orbit $G(x)$, $x\in M_1$, is closed in M_1 with respect to the relative topology. However it is shown that it is closed in M as well, by using the constancy of the length of the tensor field $X\otimes X\otimes W$ along the orbit $G(x)$, where X is the vector field defined by the one-parameter group G. Since M is compact, $G(x)$ is compact and hence G itself is compact because G is a closed subgroup of the group $I(M_1, g_1')$ of isometries. This is a contradiction and therefore M_1 is empty.

As the next step, we take the universal Riemannian covering manifold $(\tilde{M}, \tilde{g})$ of (M, g), where $\tilde{g}$ is the induced metric from g by the projection. Then $C_0(M, g)$ acts on M as a closed subgroup of $C(\tilde{M}, \tilde{g})$. We have the following lemma.

LEMMA 1.3. *Let (M, g) be a connected Riemannian n-manifold, $n>2$, and $(\tilde{M}, \tilde{g})$ a Riemannian covering manifold. Let G be a closed essential one-parameter subgroup of $C_0(M, g)$. Then it is a closed essential subgroup of $C_0(\tilde{M}, \tilde{g})$.*

The proof is based on facts that an essential one-parameter group of conformorphism always has a fixed point and that a closed one-parameter group of $C_0(M, g)$ is essential if and only if it is not compact. The detailed proof is omitted here.

In the proof of Theorem 0, the assumption of a finite fundamental group is made use of only to prove Lemma 1.3. The rest of the proof is completely applicable to our case and furthermore, since M is compact, M is conformorphic to S^n.

We remark here that the proof above is based on [10] and a classification of essential one-parameter groups of conformorphisms on S^n [9].

2. **Conjecture II.** In this section let (M, g^*) be a connected compact Riemannian n-manifold, $n>2$, with constant scalar curvature $k^*=K^*/n(n-1)$, where $K^*=K^{*h}_{hji}g^{*ji}$ is the contracted curvature scalar. It is known [5], [7] that if there is a nonisometric conformorphism, then k^* must be positive. Therefore we may assume $k^*=1$, or $K^*=n(n-1)$, without loss of generality.

If $C_0(M, g^*)$ is essential, then, by Theorem I, (M, g^*) is conformorphic to a Euclidean n-sphere (S^n, g) of radius 1. Thus (M, g^*) is considered as S^n with Riemannian metric $g^*=e^{2\rho}g$ for some smooth function ρ on S^n. Since g^* has scalar curvature 1, if Conjecture II′ is true, so is Conjecture II.

THEOREM II′. *Conjecture* II′ *is true.*

PROOF. Any quantities formed with g^* will be denoted by the same letter with asterisk as the corresponding ones formed with g. Then by well-known general formulas for a conformal change of metrics, see for example [12], on putting $\phi=e^{-\rho}$ and $G_{ji}=K_{ji}-(K/n)g_{ji}$, we have

$$\overset{*}{G}_{ji} = G_{ji} + (n-2)P_{ji}, \tag{2.1}$$

where we have put

$$P_{ji} = \phi^{-1}\left(\nabla_j\nabla_i\phi - \frac{\Delta\phi}{n}g_{ji}\right), \tag{2.2}$$

∇_j denoting covariant differentiation with respect to g and $\Delta\phi = g^{ji}\nabla_j\nabla_i\phi$. As for scalar curvature,

$$K^* = \phi^2K + 2(n-1)\phi\Delta\phi - n(n-1)\phi_i\phi^i,$$

where $\phi_i=\nabla_i\phi$ and $\phi^i=g^{ij}\phi_j$. In our case of (S^n, g), we have $G_{ji}=0$ and $K=K^*=n(n-1)$, and hence we obtain

$$\Delta\phi = \frac{n}{2}\phi^{-1}(1 - \phi^2 + \phi_i\phi^i). \tag{2.3}$$

We have only to show that P_{ji} vanishes identically under the condition (2.3). To do this let us consider a quantity

$$A = \phi^{3-n}P_{ij}P^{ij} = \phi^{1-n}\left(\nabla_j\phi_i\nabla^j\phi^i - \frac{1}{n}(\Delta\phi)^2\right) \geqq 0$$

and a vector field $V^i = \phi^{2-n}P_{jk}g^{ki}\phi^j$.

By a straightforward manipulation we obtain $\nabla_i V^i = A$. Here we have used (2.3) and the Ricci formula $\nabla_h\nabla_i\phi^h = \nabla_i(\Delta\phi) + K_{ij}\phi^j$. Thus we have $\int A\, dS = \int \nabla_i V^i\, dS = 0$, where dS is the volume element determined by g. Since $A \geqq 0$, we have $A = 0$. Since ϕ never vanishes, we have $P_{ij} = 0$, and hence $G^*_{ij} = 0$. Thus g^* is an Einstein metric and is conformally flat. It follows that g^* has constant curvature 1. This argument is seen in [**12**, Proposition 3.3].

To see that g^* is induced from g by an element σ of $C(S^n, g)$, we consider the system of equations, $P_{ij} = 0$, i.e.

$$\nabla_j\nabla_i\phi - \frac{\Delta\phi}{n} g_{ji} = 0, \qquad \phi > 0. \tag{2.4}$$

It is known [**11**] that (2.4) is equivalent to a system

$$\nabla_j\nabla_i\psi + \psi g_{ji} = 0 \tag{2.5}$$

where $\phi = \psi + a$ for some constant a determined by (2.3). An element $\sigma \in C(S^n, g)$, with $\sigma^* g = e^{2\rho(\sigma)}g$, determines (2.4) by putting $\phi(\sigma) = e^{-\rho(\sigma)}$, and the correspondence $\sigma \to \phi(\sigma)$ is a map of $C(S^n, g)$ onto the space of solutions of (2.4).

Thus g^* above is written as $g^* = \sigma^* g$ for some $\sigma \in C(S^n, g)$.

THEOREM II. *Conjecture* II *is true.*

This is obvious because of Theorem II′.

REMARK. Theorem II′ can be naturally generalized to the case of Einstein spaces.

BIBLIOGRAPHY

1. A. Avez, *Transformations conformes des variétés riemanniennes compactes*, C. R. Acad. Sci. Paris **259** (1964), 4469–4472. MR **31** #1635.

2. B. Ba, *Structures presque complexe, structures conformes et dérivations*, Topologie et Géométrie Différentielle, vol. 8, Centre Nationale Recherche Sci., Paris, 1966. MR **37** #5809.

3. A. J. Ledger and M. Obata, *Compact riemannian manifolds with essential groups of conformorphisms*, Trans. Amer. Math. Soc. **150** (1970), 645–651.

4. J. Lelong-Ferrand, *Transformations conformes et quasi-conformes des variétés riemanniennes; application à la demonstration d'une conjecture de A. Lichnerowicz*, C. R. Acad. Sci. Paris **269** (1969), 583–586.

5. A. Lichnerowicz, *Géométrie des groupes de transformations*, Travaux et Recherches Mathematiques, III, Dunod, Paris, 1958. MR **23** #A1329.

6. T. Nagano, *On conformal transformations of Riemannian spaces*, J. Math. Soc. Japan **10** (1958), 79–93. MR **22** #1859.

7. M. Obata, *Conformal transformations of compact Riemannian manifolds*, Illinois J. Math. **6** (1962), 292–295. MR **25** #1507.

8. ———, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340. MR **25** #5479.

9. ———, *Conformal transformations of Riemannian manifolds*, J. Differential Geometry **4** (1970), 311–333.

10. ———, *Conformally flat Riemannian manifolds admitting a one-parameter group of conformal transformations*, J. Differential Geometry **4** (1970), 335–337.

11. K. Yano and M. Obata, *Sur le groupe de transformations conformes d'une variété de Riemann dont le scalaire de courbure est constant*, C. R. Acad. Sci. Paris **260** (1965), 2698–2700. MR **31** #697.

12. ———, *On conformal changes of Riemannian metrics*, J. Differential Geometry **4** (1970), 53–72.

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