A CHARACTERIZATION THEOREM FOR CELLULAR MAPS

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Introduction. The main result of this paper is that a mapping f of the n-sphere ∂B^{n+1}, $n \neq 4$, onto itself is cellular if and only if f has a continuous extension which maps the interior of the $n+1$ ball B^{n+1} homeomorphically onto itself. Since a map of a 2-sphere onto itself is cellular if and only if it is monotone, this theorem extends a result of Floyd and Fort [6], who prove the corresponding theorem for monotone maps on a 2-sphere.

Preliminaries. A compact mapping $f: M^n \to X$ is cellular if for each $x \in X$, there is a sequence $C_1, C_2, \cdots$ of topological n-cells such that $f^{-1}(x) = \bigcap_{i=1}^{\infty} C_i$ and $C_{i+1} \subset \operatorname{Int} C_i$. If X is a topological space, $H(X)$ is the group of all homeomorphisms of X onto itself. Edwards and Kirby showed that for any compact manifold M, $H(M)$ is locally contractible and therefore uniformly locally arcwise connected. It was shown [7] that any mapping of a manifold onto itself which can be uniformly approximated by homeomorphisms is cellular. (See also [4].) Armentrout ($n=3$) [1] and Siebenmann ($n \geqq 5$) [10] have proven that any cellular mapping of a manifold onto itself can be uniformly approximated by homeomorphisms.

LEMMA 1. *Suppose $f: \partial B^n \to \partial B^n$ can be approximated by homeomorphisms. Then f can be extended to a map which is a homeomorphism on the interior of B^n.*

PROOF. Since f can be uniformly approximated by homeomorphisms and $H(\partial B^n)$ is uniformly arcwise connected, there is an arc Φ such that $\Phi_1 = f$ and $\Phi_t \in H(\partial B^n)$, for $0 \leqq t < 1$. Each point of B^n can be represented in the form tx, where $x \in \partial B^n$ and $0 = t = 1$. We define $F: B^n \to B^n$ by $F(tx) = t\Phi_t(x)$, for all $x \in \partial B^n$. We note that F is continuous, extends f and is a homeomorphism when restricted to the interior of B^n.

Therefore, if $n \neq 4$ and $f: \partial B^{n+1} \to \partial B^{n+1}$ is cellular f can be extended to a map which is a homeomorphism on the interior of B^{n+1}.

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A map has property UV^∞ if for each x and each open set U containing $f^{-1}(x)$, there is an open set V containing $f^{-1}(x)$ such that $V \subset U$ and V is null-homotopic in U.

LEMMA 2. *Let M be a manifold and $F: M\times(0, 1] \to M\times(0, 1]$ be a map such that $F^{-1}(M\times 1) = M\times 1$ and $F/M\times(0, 1): M\times(0, 1) \to M\times(0, 1)$ is a homeomorphism, then $F/M\times 1: M\times 1 \to M\times 1$ is a* UV^∞ *map.*

PROOF. We identify M with $M\times 1$. We make use of the following auxiliary maps: for each ∂, define $\pi_\partial: M\to M\times(1-\partial)$ by $\pi_\partial(x) = (x, 1-\partial)$ and $p: M\times(0, 1]\to M$ by $p(x, t) = (x, 1) = x$.

Let U' be open in M with $f^{-1}(b)\subset U'$. $U'\times(0, 1]$ is open in $M\times(0, 1]$. Therefore, there is a U such that:

(a) U is open in $M\times(0, 1]$.

(b) $U\subset U'\times(0, 1]$.

(c) $f(U)$ is open in $M\times(0, 1]$.

(d) $F^{-1}(b)\subset U$.

Now choose $t_0 < 1$ and an open cylinder, C, about $b\times[t_0, 1]$ such that $C\subset f(U)$. We note that:

$f^{-1}(C)$ is open in $M\times(0, 1]$, $f^{-1}(C)\subset U$, $f^{-1}(b\times[t_0, 1])\subset f^{-1}(C)$.

Let $\eta = d(b, \tilde{C})$; $\eta > 0$. Let δ be chosen so that

(a) $N_{2\delta}(f^{-1}(b))\subset f^{-1}(C)$.

(b) $d(x, y) < 2\delta \Rightarrow d(f(x), f(y)) < \eta$.

Let $V = N_\delta(f^{-1}(b))\cap M$. We note that if x is an element of $\pi_\delta(V)$, then $f(x)$ is an element of $N_\eta(b)\cap M\times(0, 1)\subset C$.

Since C is a cell we can define a homotopy $G: C\times I\to C$ so that

(1) $x\in C^\circ \Rightarrow G(x, t)\in C\cap(M\times(0, 1))$.

(2) $G(x, 0) = x$.

(3) $\exists\, z\in M\times(0, 1)$ such that $G(x, 1) = z$, for all $x\in C$.

We now can define the desired homotopy $H: V\times I\to U'$, by $H(x, t) = pf^{-1}(G(f\pi_\delta(x), t))$. Thus, $H(x, 0) = pf^{-1}[G(f\pi_\delta(x), 0)] = pf^{-1}(f\pi_\delta(x)) = x$.

$$H(x, 1) = pf^{-1}[G(f\pi_\delta(x), 1)] = pf^{-1}(z) = \text{constant}.$$

The continuity of f follows from that of G, so all that remains to be shown is that $H(x, t)\in U'$, for all $x\in V$, $\forall\, t\in I$.

$$x \in V \Rightarrow \pi_\delta(x) \in \pi_\delta(V) \Rightarrow f(\pi_\delta(x)) \in C \cap M \times (0, 1)$$
$$\Rightarrow G(f\pi_\delta(x), t) \in C \cap B^\circ \Rightarrow$$

that f^{-1} is defined and $f^{-1}[G(f\pi_\delta(x), t)]\in f^{-1}(C)\subset U\subset U'\times(\frac{1}{2}, 1]$. Thus $P(f^{-1}[G(f\pi_\delta(x), t)]) = H(x, t)\in U'$.

Let $M \subset X$. M is collared if there is a homeomorphism $h: M \times (0, 1] \to$nbd of M such that $h(m, 1) = m$, for all $m \in M$. M. Brown proved that the boundary of any manifold with boundary is collared [3]. Therefore, we have the following corollary.

COROLLARY. *Let M be a manifold with boundary and let $f: M \to M$ be such that f restricted to the interior of M is a homeomorphism. Then $f/\partial M$ is a* UV$^\infty$*-map.*

Using McMillan's criteria for cellularity, [9] it can easily be shown that if $f: M^n \to M^n$ is a UV$^\infty$-map and if $M^n = S^3$ or $n \geqq 5$, then f is a cellular map. (Cf., Armentrout and Price [2] or Lacher [8].) We therefore have the following theorem:

THEOREM. *A mapping f of the n-sphere ∂B^{n+1}, $n \neq 4$, onto itself is cellular iff f has a continuous extension which maps the interior of B^4 homeomorphically onto itself.*

COROLLARY. *Let M be an m-manifold, $n \geqq 5$, with boundary. Let f be a map of M onto M such that $f/\mathrm{Int}\ M: \mathrm{Int}\ M \to \mathrm{Int}\ M$ is cellular and $f/\partial M: \partial M \to \partial M$. Then $f/\partial M$ is a* UV$^\infty$ *map. In particular, if $n \geqq 6$, f/M is a cellular map.*

PROOF. Define $g: \mathrm{Int}\ M \to (0, \infty)$ by $g(m) = d(m, \partial M)$. Since $f/\mathrm{Int}\ M$ is a cellular map, by Siebenmann's theorem there is a homeomorphism h such that for all $x \in \mathrm{Int}\ M$, $d(f(x), h(x)) < g(f(x))$. We define $F: M \to M$ by

$$\begin{aligned} F(x) &= h(x), \qquad x \in \mathrm{Int}\ M, \\ &= f(x), \qquad x \in \partial M. \end{aligned}$$

F is continuous, for suppose there is a sequence, x_n, of points in Int M which converge to $x \in \partial M$. Let $\epsilon > 0$ be given. By the continuity of f, $\exists\ N \ni n > N \Rightarrow d(f_n(x), f(x)) < \epsilon/2$. Then for such n,

$$d(F(x_n), F(x)) = d(h(x_n), f(x)) \leqq d(h(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon.$$

Thus, by Lemma 2, $F/\partial M = f/\partial M$ is a UV$^\infty$ map.

Armentrout's approximation theorem [1] and results of E. E. Floyd [5] make it possible to prove the corresponding result for three manifolds: For such M, if $f: M \to M$ is a proper map such that $f/\mathrm{Int}\ M$ is cellular, then $f/\partial M$ is cellular.

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