HOLOMORPHIC FUNCTIONS ON A BANACH SPACE

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1. Unless otherwise stated we shall use the definitions and notation of [4]. E will always be a complex Banach space and $\mathcal{H}(E)$ will denote the set of all complex valued holomorphic functions on all of E. θ will be a holomorphy type [4, p. 34] and $(\mathcal{H}_\theta(E), \mathfrak{I}_{\omega,\theta})$ will denote the topological vector space of holomorphic functions associated with θ as described in [4, pp. 35, 43]. c_0^+ will denote the set of all sequences of positive real numbers which tend to zero at infinity. $\mathcal{K}$ will denote the set of all convex balanced compact subsets of E. Let B_1 be the closed unit ball of some fixed norm which defines E. $\| \ \|_U$ will denote the Minkowski functional of the subset U of E.

An α-holomorphy type θ is a holomorphy type whose definition depends only on the topological vector space structure of E (i.e., it is independent of the actual norm used to define E) and if U, $V \subset E$ and C is a positive real number such that $CU \subset V$ then for each n we have

$$C^n \|P_n\|_{\theta,U} \leqq \|P_n\|_{\theta,V} \qquad \text{for all } P_n \in \mathcal{P}_\theta(^nE)$$

where $(\mathcal{P}_\theta(^nE), \| \ \|_\theta)$ is the $(n+1)$st member of the sequence given by θ (see [4, p. 34]).

DEFINITION 1. (a) *Let θ be an α-holomorphy type then $H_\theta(E)$ is the set of all elements of $\mathcal{H}(E)$ for which*

(1) $\hat{d}^n f(0) \in \mathcal{P}_\theta(^nE)$.

(2) *For each $K \in \mathcal{K}$, $\exists \epsilon > 0$ such that*

$$\sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, K+\epsilon B_1} < \infty.$$

(b) *A seminorm p on $H_\theta(E)$ is θ-ported by $K \in \mathcal{K}$ if, for each $\epsilon > 0$, $\exists C(\epsilon) > 0$ such that*

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$$p(f) \leqq C(\epsilon) \sum_{m=0}^{\infty} \left\| \frac{\hat{d}^m f(0)}{m!} \right\|_{\theta, K+\epsilon B_1} \qquad \textit{for all } f \in H_\theta(E).$$

(c) *The topology T_θ on $H_\theta(E)$ is that generated by all seminorms θ-ported by some element of $\mathcal{K}$.*

For the nuclear type this is the topology considered in [6].

THEOREM 1. (a) *Let $f = \sum_{n=0}^{\infty} (\hat{d}^n f(0)/n!) \in \mathcal{H}(E)$ and $\hat{d}^n f(0) \in \mathcal{P}_\theta(^nE)$ for $n = 0, 1, \cdots$. Then the following conditiions are equivalent:*

(1) $f \in H_\theta(E)$.

(2) *For each $K \in \mathcal{K}$, $(\alpha_n)_{n=0}^{\infty} \in c_0^+$, we have*

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\alpha_n B_1} < \infty.$$

(3) *For each $K \in \mathcal{K}$, $(\alpha_n)_{n=0}^{\infty} \in c_0^+$, we have*

$$\lim_{n \to \infty} \left\| \frac{d^n f(0)}{n!} \right\|_{\theta, K+\alpha_n B_1}^{1/n}.$$

(b) *The T_θ topology on $H_\theta(E)$ is generated by all seminorms of the form*

$$p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\theta, K+\alpha_n B_1}$$

as K ranges over $\mathcal{K}$ and $(\alpha_n)_{n=0}^{\infty}$ ranges over c_0^+.

(c) *$(H_\theta(E), T_\theta)$ is a complete topological vector space.*

Taking θ as the current type we get a global definition for elements of $\mathcal{H}(E)$.

THEOREM 2. *Let $(P_n)_{n=0}^{\infty}$ be a sequence of continuous polynomials, P_n being homogeneous of degree n, then $\sum_{n=0}^{\infty} P_n$ is the Taylor series expansion of an element of $\mathcal{H}(E)$ if and only if for each $K \in \mathcal{K}$ and $(\alpha_n)_{n=0}^{\infty} \in c_0^+$ we have*

$$\lim_{n \to \infty} \|P_n\|_{K+\alpha_n B_1}^{1/n} = 0.$$

REMARKS. (1) We do not know if $(H_\theta(E), T_\theta)$ is a bornological space. However we can get the following result:

The bornological topology on $H_\theta(E)$ associated with T_θ is the finest locally convex topology on $H_\theta(E)$ for which the Taylor series converges absolutely and which induces on each space $\mathcal{P}_\theta(^nE)$ its original norm topology.

(2) For any α-holomorphy type $(H_\theta(E), T_\theta) \subset (\mathcal{H}_\theta(E), \mathfrak{I}_{\omega,\theta})$ continuously. For the current type θ we get $H_\theta(E) = \mathcal{H}_\theta(E) = \mathcal{H}(E)$ and $T_\theta = \mathfrak{I}_{\omega,\theta}$.

2. We denote by N the nuclear type which in a certain sense (see [7]) can be looked upon as a maximal type.

DEFINITION 2. *An α-holomorphy type θ is an α-β-holomorphy type if*

(1) *For each n, $\mathcal{P}_\theta({}^nE) \supset \mathcal{P}_N({}^nE)$ and $\|P_n\|_{\theta,U} \leqq \|P_n\|_{N,U}$ for all $P_n \in \mathcal{P}_N({}^nE)$, and all $U \subset E$.*

(2) *For each n, $\mathcal{P}_f({}^nE)$ (see [5]) is a dense subspace of $(\mathcal{P}_\theta({}^nE), \| \ \|_\theta)$.*

For α-β-holomorphy types we get a Banach space isomorphism between $(\mathcal{P}_\theta({}^nE), \| \ \|_\theta)'$ and a subspace of $\mathcal{P}({}^nE')$ which we denote by $(\mathcal{P}_{\theta'}({}^nE'), \| \ \|^{\theta'})$.

The Borel transform of $T \in (H_\theta(E), T_\theta)'$, $\hat{T}$, is a function defined on E' by $\hat{T}(\phi) = T(\exp \phi)$. The Borel transform gives a one-to-one correspondence between elements of $(H_\theta(E), T_\theta)'$ and a vector space of holomorphic functions on E' which we shall call the θ'-compact exponential type in E.

To consider partial differential operators we put further conditions on α-β-holomorphy types and get what we call α-β-γ-holomorphy types. The conditions are for the most part natural and one is a slightly weaker form of the requirement that $(\mathcal{P}_{\theta'}({}^nE'), \| \ \|^{\theta'})_{n=0}^{\infty}$ be a holomorphy type. We then define the Fréchet space of θ-formal power series, $\mathfrak{F}\mathcal{P}_\theta(E)$.

THEOREM 3. *If θ is an α-β-γ-holomorphy type and Q is a partial differential operator on $\mathfrak{F}\mathcal{P}_\theta(E)$ then*

(1) *All solutions of Q (i.e., $\forall f$ such that $Q(f) = 0$) can be approximated in $\mathfrak{F}\mathcal{P}_\theta(E)$ by θ-exponential polynomial solutions.*

(2) *If $Q \neq 0$ then Q maps $\mathfrak{F}\mathcal{P}_\theta(E)$ onto itself.*

We will publish in a future paper full proofs of the above theorems and assertions together with some related results and examples.

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