SURFACES OF VERTICAL ORDER 3 ARE TAME

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We define a 2-sphere S in E^3 to have *vertical order n* if each vertical line intersects S in no more than n points. The main result in this paper is the following

THEOREM 1. *If S is a 2-sphere in E^3 having vertical order* 3, *then S is tame.*

This is the best theorem possible in the sense that examples are known of wild 2-spheres in E^3 having vertical order 4 [**5**]. In Theorem 2 to follow we generalize Theorem 1 to compact 2-manifolds in E^3.

Previous work concerned with the nature of the intersection of vertical lines with a 2-sphere in E^3 has been done by Bing [**1**, Theorem 7.3]; [**3**].

PROOF OF THEOREM 1. The vertical line in E^3 containing the point x is denoted by L_x, and we refer to the bounded component of E^3-S as Int S. If $x\in\operatorname{Int} S$ it is easy to see that $L_x\cap S$ consists of two points. In this case the point with largest third coordinate is denoted by U_x and the other by V_x. We let $U=\{U_x | x\in \operatorname{Int} S\}$ and $V=\{V_x | x\in\operatorname{Int} S\}$, and we note that U and V are both open subsets of S. A bicollar can be constructed for a neighborhood of each point of $U\cup V$ using short vertical intervals. Thus S is locally tame at each point of $U\cup V$ [**2**].

Let $R=S-(U\cup V)$. The proof that S is tame is completed by showing that R is a tame simple closed curve, since a 2-sphere that is locally tame modulo a tame simple closed curve is known to be tame [**4**].

It will follow that R is a simple closed curve once we show that each of U and V is connected and that each point $p\in R$ is arcwise accessible from both U and V [**7**, p. 233]. Let θ be an arc in Int S $\cup\{p\}$ such that p is an endpoint of θ. We now show that the vertical projection σ of θ into $U\cup\{p\}$ is continuous. To accomplish this we take a sequence $\{x_i\}$ of points in θ converging to x_0 and we prove that the sequence $\{\sigma(x_i)\}$ converges to $\sigma(x_0)$. Let L_i $(i=0, 1, 2, \cdots)$ be the vertical interval from x_i to $\sigma(x_i)$ (if $x_i=p$, then L_i is degenerate),

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and let L = limit superior $\{L_i\}$. Then L is an interval (possibly degenerate) in L_{x_0} having lower endpoint x_0. The upper endpoint of L must lie in S so it cannot lie below $\sigma(x_0)$. On the other hand L cannot properly contain L_0 either, since then there would be limit points of Int S in Ext S. Thus $L=L_0$, and we see that $\{\sigma(x_i)\}$ converges to $\sigma(x_0)$. Since $\sigma(\theta)$ is the continuous image of an arc, it must contain an arc having p as an endpoint. Now p is arcwise accessible from V by the same reasoning. A similar argument would establish the arcwise connectivity of both U and V, so we may conclude that R is a simple closed curve.

All that remains is to show that R is tame. We let $p\in R$ and we let $\epsilon>0$. In the remainder of the proof we construct a 2-sphere S' such that $p\in \text{Int } S'$, S' has diameter less than ϵ, and $S'\cap R$ consists of two points. This is enough to ensure the tameness of R since it implies that R satisfies property P of [6].

Let N be a round open neighborhood of p with diameter less than ϵ, and let α and β be two arcs in R such that $\alpha\cap\beta=\{p\}$, $\alpha\cup\beta\subset N$, and no point of $N\cap R$ lies vertically above p. There is an arc γ joining the two endpoints of $\alpha\cup\beta$ such that Int $\gamma\subset U$ and $\alpha\cup\beta\cup\gamma$ is a simple closed curve J bounding an open disk W in U. Since the vertical projection π of E^3 onto a horizontal plane E^2 is continuous, there exist arcs α' and β' in $\pi(\alpha)$ and $\pi(\beta)$, respectively, such that $\pi(\text{Bd}\alpha)=\text{Bd}\alpha'$ and $\pi(\text{Bd}\beta)=\text{Bd}\beta'$. The proof is completed in two cases.

Case 1. $\alpha'\cap\beta'$ is nondegenerate. In this case we let x' be a point of $\alpha'\cap\beta'$ such that $x'\neq\pi(p)=p'$, and we let $x\in\alpha$ and $y\in\beta$ be two points such that $\pi(x)=\pi(y)=x'$. Let f be an arc from x to y such that $f-\{x, y\}\subset W$, and notice that $f'=\pi(f)$ is a simple closed curve. It is not difficult to show that p lies in Int H where $H=\pi^{-1}(f')$ and Int H is the component of E^3-H whose intersection with E^2 is bounded. We form a 2-sphere T in $\overline{N}$ by taking the union of $H\cap N$ with the two disks in (Bd N)$\cap$($H\cup$Int H). If $T\cap R=\{x, y\}$ we let $T=S'$. Otherwise $T\cap R$ consists of three points and it follows that R cannot pierce T at all three points. Thus R must be tangent to H at one point, and we may move T slightly to the nontangency side of H near the nonpiercing point to form S' in this case.

Case 2. $\alpha'\cap\beta'=\{p'\}$. Let $\gamma'=\pi(\gamma)$ and $W'=\pi(W)$. In this case $\alpha'\cup\beta'\cup\gamma'$ is a simple closed curve J' bounding a disk D' in E^2. It is not difficult to see that $W'\subset\text{Int } D'$ because $W'\cap J'=\emptyset$ and W' is arcwise connected. Let N_1 and N_2 be round open neighborhoods of p such that $\overline{N}_1\cap R\subset\alpha\cup\beta$ and each pair of points of $R\cap N_2$ lies in an arc in $(\alpha\cup\beta)\cap N_1$. If $x'\in\alpha'\cap\pi(N_2)$ and $y'\in\beta'\cap\pi(N_2)$ there is an arc g' from x' to y' such that $g'-\{x', y'\}\subset E^2-D'$ and $g'\subset\pi(N_1)$.

The idea of the remainder of the proof is to obtain an arc f' from x' to y' such that $f' - \{x', y'\} \subset \pi(N_1) \cap W'$, and then to construct the 2-sphere S' using part of the infinite vertical cylinder $H = \pi^{-1}(f' \cup g')$ and two disks in Bd N_1. Of course, this requires a nice enough selection of x' and y' to insure that H intersects R in a controlled manner.

Suppose we are able to select $x' \in \alpha'$ and $y' \in \beta'$ each having exactly one point, say x and y respectively, of $\alpha \cup \beta$ vertically above it. Then an arc f can be constructed in $N_1 \cap (W \cup \{x, y\})$ whose projection $\pi(f)$ satisfies the desired conditions on f', and it would follow that $H \cap (\alpha \cup \beta) = \{x, y\}$. Thus S' could be chosen as the boundary of the 3-cell $(H \cup \text{Int } H) \cap \overline{N}_1$, and it would follow that $R \cap S' = \{x, y\}$. We show now that such points x' and y' can always be found.

Suppose that for each $x' \in \beta'$ the set $\pi^{-1}(x') \cap \beta$ contains at least two points. We can select $x' \in \beta'$ such that $\pi^{-1}(x') \cap \beta$ contains two points x_1 and x_2 having the property that every open arc in β with either x_1 or x_2 as an endpoint intersects $\pi^{-1}(\beta')$. This is possible because $\beta - \pi^{-1}(\alpha' \cup \beta')$ has at most countably many components, and any point not in the projection of their endpoints would satisfy the conditions on x'.

We choose disjoint arcs A_1 and A_2 such that $x_i \in \text{Int } A_i \subset A_i \subset \beta$ and $A_i \cap L_{x_i} = \{x_i\}$ $(i = 1, 2)$, and we choose disjoint disks D_1 and D_2 such that $A_i \subset \text{Bd } D_i$, $(D_i - A_i) \subset W$, and Bd $D_i - A_i$ is an open arc B_i in W $(i = 1, 2)$. In view of the selection of x' we may assume that the endpoints of each A_i lie in $\pi^{-1}(\beta')$. This implies that the open arcs $B_i' = \pi(B_i)$ have their endpoints in β'. Notice that $B_1' \cap B_2' = \emptyset$, for otherwise a vertical line through a point of $B_1' \cap B_2'$ would intersect W twice; and recall that $\pi(D_1 \cup D_2) \subset \pi(W \cup J) \subset D'$. Since x' lies in the boundary of both $\pi(D_1)$ and $\pi(D_2)$, we see that the endpoints of each B_i' separate x' from γ' in Bd D'. Thus the closure of one of B_1' and B_2', say B_1', separates the other from γ in D'. This forces $\overline{B}_2'$ to separate B_1' from x' in D', and yields a contradiction since there is an arc in D_1 from x_1 to a point of B_1 missing B_2.

THEOREM 2. *If S is a compact 2-manifold in E^3 having vertical order 3, then S is tame.*

We restrict ourselves here to an outline of the proof of Theorem 2. By working with a component of S we may suppose that S is connected and consequently that S has exactly two complementary domains. The sets U, V, and R are defined just as in the proof for Theorem 1, and in the same way we see that U and V are connected, open, and locally tame. In this case R is a finite collection of disjoint simple closed curves each of which can be proven tame by establishing

Properties P and Q of [6] as before. Thus S is tame, since it is locally tame modulo a finite collection of tame simple closed curves.

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