COMPACT FAMILIES OF ALMOST PERIODIC FUNCTIONS

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A family of almost periodic functions on the reals, R, to complex n-space, C^n, is compact in the uniform topology if and only if it is (a) closed, (b) uniformly bounded, (c) uniformly equicontinuous, and (d) uniformly almost periodic. This is a result of Bochner [1]. Of the above criteria, part (d) seems to be the most difficult to verify. We offer two results in this direction.

Recall that the family A is uniformly almost periodic if for each $\epsilon>0$, the set $T(A, \epsilon)\equiv\bigcap_{f\in A}\{\tau: |f(x+\tau)-f(\tau)|<\epsilon \text{ for all } x\in R\}$ is relatively dense. For A a singleton, this is Bohr's definition of an almost periodic function. Let $\exp(\phi)$ be the set of real numbers λ, such that $\lim_{T\to\infty}(1/T)\int_0^T\phi(x)e^{-i\lambda x}dx\neq 0$. If A is a compact family then $\exp(A)=\bigcup_{f\in A}\exp(f)$ is countable. Hence we will consider sets of the following form. Let $C(M,\Lambda)\equiv\{\phi\,|\,\phi \text{ is almost periodic}, \|\phi\|\leqq M, \exp(\phi)\subset\Lambda\}$ where M is a fixed real number, $\|\cdot\|$ is the supremum norm, and Λ is a given countable set of reals.

THEOREM 1. *If Λ has no finite limit point, then any uniformly equicontinuous family in $C(M,\Lambda)$ has compact closure.*

THEOREM 2. *If $A\subset C(M,\Lambda)$ is the family that is uniformly Lipschitz, i.e.: there is a $K>0$, such that $f\in A$ if and only if $|f(t)-f(s)|\leqq K|t-s|$ for all t and s, then A has compact closure if and only if Λ has no finite limit point.*

In fact, if Λ has no finite limit point, then A is a convex compact set having the fixed point property.

If Λ has no finite limit point, then let $\Lambda=\{\lambda_n\}$ with $|\lambda_1|\leqq|\lambda_2|\leqq\cdots$. By a result of Bredhina [2], there exist polynomials $\sigma_n(f,x)=\sum_{k=1}^n a_k(f)e^{i\lambda_k x}$ such that $\|f-\sigma_n(f)\|\leqq 10\;\omega_f(1/|\lambda_n|)$ where $\omega_f(x)=\sup\{|f(y)-f(z)|:|y-z|\leqq x\}$. For any $\epsilon>0$, the polynomials $\{\sigma_{n_0}(f)\}_{f\in A}$ are an ϵ-net for some n_0. This collection is uniformly almost periodic; hence so is A.

If the family A is Lipschitz and Λ has a finite limit point, one constructs a closed ball of an infinite dimensional space in A. These are not compact. One uses the result of Bochner [3] to the effect that if $\exp(\phi)$ is a bounded set, then ϕ' exists and $\|\phi'\|\leqq T\|\phi\|$ where T is a bound for $\exp(\phi)$. That is, for such functions, the Lipschitz condition is automatically satisfied.

Details of the above outline appear in [4] where an application of the fixed point property is also given.

References

1. S. Bochner, *Beiträge zur Theorie der fastperiodischen Funktionen*. I. *Funktionen einer variablen*, Math. Ann. **96** (1926), 119–147.

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3. S. Bochner, *Properties of fourier series of almost periodic functions*, Proc. London Math. Soc. **26** (1927), 433–452.

4. A. M. Fink, *Compact families of almost periodic functions and an application of the Schauder fixed point theorem*, SIAM J. Appl. Math. (to appear).

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