A GENERALIZATION OF THE AHLFORS-HEINS THEOREM[1]

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Let D be the complex plane cut along the negative real axis. We are going to consider a function u subharmonic in D. Let $M(r) = \sup_{|z|=r} u(z)$ and $m(r) = \inf_{|z|=r} u(z)$. We also introduce, for $r>0$, $v(r) = \limsup_{z\to -r+i0} u(z)$, $\bar{v}(r) = \limsup_{z\to -r-i0} u(z)$ and $u(-r) = \max(v(r), \bar{v}(r))$. In the whole paper, $z = re^{i\theta}$. Our main result is

THEOREM 1. *Let* λ *be a number in the interval* (0, 1) *and let* u $(\not\equiv -\infty)$ *be a function subharmonic in* D *that satisfies*

$$u(-r) - \cos \pi\lambda\, u(r) \leqq 0. \tag{1}$$

Then either $\lim_{r\to\infty} r^{-\lambda} M(r) = \infty$ *or*

(A) *there exists a number* α *such that*

$$\lim_{r\to\infty} r^{-\lambda} u(re^{i\theta}) = \alpha \cos \lambda\theta, \qquad |\theta| < \pi, \tag{2}$$

except when θ *belongs to a set of logarithmic capacity zero.*

(B) *Given* θ_0, $0<\theta_0<\pi$, *there exists an* r-*set* Δ_0 *of finite logarithmic length such that* (2) *holds uniformly in* $\{z \mid |\theta| \leqq \theta_0\}$ *when* r *is restricted to lie outside of* Δ_0.

REMARK. When $1/2 < \lambda < 1$, condition (1) is interpreted in the following way at points where $u(-r) = \infty$.

$$\limsup_{z\to r}(u(x+iy) + u(-x+iy)) \leqq (1+\cos \pi\lambda) u(r), \tag{1a}$$

$$\limsup_{z\to r}(u(-x+iy) - \cos \pi\lambda\, u(x+iy)) \leqq 0. \tag{1b}$$

Theorem 1 can be compared to the main result of Kjellberg [6].

THEOREM 2. *Let* u *be subharmonic in the complex plane and let* λ *be a number in the interval* (0, 1). *If* $m(r) - \cos \pi\lambda\ M(r) \leqq 0$, *then the (possibly infinite) limit* $\lim_{r\to\infty} r^{-\lambda} M(r)$ *exists.*

In order to clarify the connection between Theorem 1 and the Ahlfors-Heins theorem [1], we also state Theorem 1 in the following equivalent way.

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THEOREM 3. *Let* λ *be a number in the interval* (0, 1) *and let* u *be a function subharmonic in* $\{\operatorname{Re} z>0\}$. *If for* t *real*

$$u(it) \equiv \limsup_{z\to it} u(z) \leqq \cos \pi\lambda \, u(|t|), \tag{3}$$

then either $\limsup_{r\to\infty} r^{-2\lambda}M(r)=\infty$ *or* $\lim_{z\to\infty} u(z)/(r^{2\lambda}\cos 2\lambda\theta)$ *exists in the sense of* (A) *and* (B).

If we choose $\lambda=1/2$, we obtain the Ahlfors-Heins theorem.

The proof of Theorem 1 is long and will appear elsewhere. In this announcement, we give an outline of the proof of Theorem 3 in the simpler case $0<\lambda<1/2$. In the proof, two lemmas (Lemmas 3 and 4) on convolution inequalities are used. These are stated at the end of the paper.

PROOF OF THEOREM 3 IN THE CASE $0<\lambda<1/2$. It is an unessential restriction to assume that u is harmonic, bounded and has a negative upper bound in a neighborhood of the origin.

LEMMA 1. *Under the assumptions of Theorem* 3, *either* $\lim_{r\to\infty} r^{-2\lambda}M(r)=\infty$ *or* $\limsup_{r\to\infty} r^{-2\lambda}u(r)<\infty$.

PROOF OF LEMMA 1. We apply Poisson's formula for a semicircle (cf., e.g., Boas [2, 1.2.3]). Using (3), we deduce

$$r^{-2\lambda}u^+(r) \leqq \int_0^R t^{-2\lambda}u^+(t)L(r,t)dt + \text{const.}(r/R)^{1-2\lambda}(M(R)/R^{2\lambda}),$$

where

$$L(r,t) = \frac{2\cos\pi\lambda}{\pi}\cdot\frac{(t/r)^{2\lambda}\cdot r}{t^2+r^2}$$

and

$$u^+ = \max(u, 0).$$

Since $\int_0^\infty L(r,t)dt=1$, we obtain $\sup_{0<r<R} r^{-2\lambda}u^+(r) \leqq \text{const.}\, R^{-2\lambda}M(R)$ from which Lemma 1 follows.

In the remaining part of the paper, we assume that the second alternative of Lemma 1 is valid. In particular, we have

$$\liminf_{r\to\infty} r^{-2\lambda}M(r) < \infty.$$

Letting $R\to\infty$ in the formulas used in the proof of Lemma 1, we deduce

$$r^{-2\lambda}u(r) \leqq \int_0^\infty L(r, t)t^{-2\lambda}u(t)dt. \tag{4}$$

We define $\alpha = \limsup_{r\to\infty} r^{-2\lambda}u(r)$ and $u_1(z) = u(z) - \alpha r^{2\lambda}\cos 2\lambda\theta$.

LEMMA 2. *α is finite and u_1 is a nonpositive function on the positive real axis.*

PROOF. By the change of variables $r = e^x$, $t = e^y$ in (4), we obtain a convolution inequality

$$\phi - \phi * L \leqq 0,$$

where

$$L(x) = \frac{2\cos \pi\lambda}{\pi} \frac{e^{(1-2\lambda)x}}{e^{2x}+1}$$

and $\phi(x) = e^{-2\lambda x}u(e^x)$. If α is finite, the lemma follows by applying Lemma 3 to $(\phi - \alpha)^+$. The case $\alpha = -\infty$ is treated in a similar way.

From now on, we can assume that $\limsup_{r\to\infty} r^{-2\lambda}u(r) = 0$ and that u is nonpositive on the positive real axis (if this is not true, replace u by u_1). It follows from (3) that the function $t \rightsquigarrow u(it)$, $t \in \mathbf{R}$, is also nonpositive. We define

$$w(z) = \frac{r\cos\theta}{\pi}\int_{-\infty}^{\infty} \frac{u(it)}{r^2 - 2tr\sin\theta + t^2}\,dt,$$

the integral being absolutely convergent. Applying the Phragmén-Lindelöf theorem (cf., e.g., Heins [5, p. 111]), we conclude that w is a harmonic majorant of u in $\{\operatorname{Re} z > 0\}$. The nonnegative, superharmonic function q is defined by $q = w - u$. Once more applying (3), we obtain

$$w(r) \leqq \frac{2r\cos\pi\lambda}{\pi}\int_0^\infty \frac{u(t)}{t^2+r^2}\,dt = \frac{2r\cos\pi\lambda}{\pi}\int_0^\infty \frac{w(t) - q(t)}{t^2+r^2}\,dt.$$

Since q is nonnegative, the same change of variables as in the proof of Lemma 2 gives that the function ψ defined by $\psi(x) = e^{-2\lambda x}w(e^x)$, $x \in \mathbf{R}$, is a solution of a convolution inequality. Applying Lemma 4, we obtain that $\lim_{x\to\infty}\psi(x) = \lim_{r\to\infty} r^{-2\lambda}w(r) = 0$ and that $\int_0^\infty t^{-1-2\lambda}q(t)dt$ is convergent. It is now easy to prove that $\lim_{z\to\infty} w(z)/(r^{2\lambda}\cos 2\lambda\theta) = 0$ uniformly in each inner sector of $\{\operatorname{Re} z > 0\}$. It remains for us to consider q.

We claim that $\lim_{z\to\infty} q(z)/(r^{2\lambda}\cos 2\lambda\theta) = 0$ in the sense of (A) and (B). It is an unessential restriction to assume that $\lim_{z\to it} q(z) = 0$,

$t \in \boldsymbol{R}$ (if this is not true, replace $-q$ by $\max(-q, -x)$). Let the subharmonic function h_1 be $-q$ in the fourth quadrant and the least harmonic majorant of $-q$ in the first quadrant. Using repeated harmonic continuations and conformal mappings, we construct a function subharmonic in a half-plane which fulfills the assumptions of the Ahlfors-Heins theorem. The essential properties of the exceptional sets are not changed by conformal mappings, and going back to h_1, we obtain our result in the fourth quadrant. Interchanging the role of the quadrants in the previous construction, we obtain the existence of the limit in $\{\text{Re } z>0\}$, and the proof of Theorem 3 in the case $0<\lambda<1/2$ is complete.

An alternative way of stating Theorem 3 is to use the concept of fine topology (cf. Doob [**3**] for references). It is worth mentioning that in the case $0<\lambda<1/2$, our assumptions imply that $u(z)$ has a finite fine limit almost everywhere on the imaginary axis. This property of u follows immediately from Theorem 4.3 of Doob [**3**], applied to the nonpositive subharmonic function u and the positive harmonic function $z \leadsto r^{2\lambda} \cos 2\lambda\theta$, $\text{Re } z>0$.

Finally, we state the lemmas on convolution inequalities. They are variations on the result of Essén [**4**]. For simplicity, we only consider the kernel L mentioned in the proof of Lemma 2, and study the convolution inequality

$$\phi - \phi * L \leqq 0. \tag{5}$$

A solution of (5) is a locally integrable function ϕ such that $\phi*L$ converges absolutely and (5) is true.

LEMMA 3. *Let ϕ be a bounded solution of* (5). *If* $\lim_{|x|\to\infty} \phi(x)=0$, *then* $\phi(x)=0$ *a.e.*

We define

$$\begin{aligned} \phi_c(x) &= \phi(x), \quad && \phi(x) \geqq -c, \\ &= -c, && \phi(x) < -c. \end{aligned}$$

LEMMA 4. *Let ϕ be a nonpositive solution of* (5). *If* $\limsup_{x\to\infty} \phi(x) =0$, *then* $\phi-\phi*L \in L^1(0, \infty)$. *If furthermore there exists a positive constant c such that ϕ_c is slowly decreasing at infinity* (*cf.* [**7**, *Chapter* IV (9b)]), *then* $\lim_{x\to\infty} \phi(x)=0$.

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