ON GENERALIZED COMPLETE METRIC SPACES

BY CALVIN F. K. JUNG

Communicated by J. B. Diaz, September 19, 1968

1. **Introduction.** In reference [3], Luxemburg defined the notion of generalized complete metric space as follows.

DEFINITION. The pair (X, d) is called a *generalized complete metric space* if X is a nonvoid set and d is a function from $X \times X$ to the extended reals satisfying the following conditions:

(D0) $d(x, y) \geqq 0$,

(D1) $d(x, y) = 0$ if and only if $x = y$,

(D2) $d(x, y) = d(y, x)$,

(D3) $d(x, y) \leqq d(x, z) + d(z, y)$,

(D4) every d-Cauchy sequence in X is d-convergent, i.e., if $\{x_n\}$ is a sequence in X such that $\lim_{m,n\to\infty} d(x_n, x_m) = 0$, then there is an $x \in X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.

Some fixed point theorems of the alternative for contractions on such spaces had been proved which include, as a special case, the fixed point theorem of Banach for contractions on complete metric spaces (see [1]). For further information and references, see references [2] and [4].

For convenience, we shall call a pair (X, d) a *generalized metric space* if all but condition D4 of the above definition are satisfied.

Let $\{(X_\alpha, d_\alpha) \mid \alpha \in \mathfrak{A}\}$ be a family of disjoint metric spaces. Then there is a natural way of obtaining a generalized metric space (X, d) from $\{(X_\alpha, d_\alpha) \mid \alpha \in \mathfrak{A}\}$ in the following manner. Let X be the union of $\{X_\alpha \mid \alpha \in \mathfrak{A}\}$. For any $x, y \in X$, define

$$\begin{aligned} d(x, y) &= d_\alpha(x, y) && \text{if } x, y \in X_\alpha && \text{for some } \alpha \in \mathfrak{A}, \\ &= +\infty && \text{if } x \in X_\alpha,\ y \in X_\beta && \text{for some } \alpha, \beta \in \mathfrak{A} \text{ with } \alpha \neq \beta. \end{aligned}$$

Clearly (X, d) is a generalized metric space. Moreover, if each (X_α, d_α) is also complete, then (X, d) is a generalized complete metric space. The main purpose of this paper is to show that the above procedure is the only way to obtain generalized (complete) metric spaces (see §2). Consequently, most of the fixed points theorems of the alternative on such spaces can be obtained from the corresponding fixed point theorems on metric spaces (see §3).

2. **Decomposition.** Let (X, d) be a generalized metric space. Define a relation $\sim$ on X as follows.

$x \sim y$ if and only if $d(x, y) < +\infty$. Then $\sim$ is obviously an equivalence relation on X and, therefore, X is decomposed uniquely into (disjoint) equivalence classes X_α, $\alpha \in \mathfrak{A}$. We shall call this decomposition of X the *canonical decomposition*, for convenience.

THEOREM. *Let (X, d) be a generalized metric space. $X = \bigcup \{X_\alpha \mid \alpha \in \mathfrak{A}\}$ the canonical decomposition and*

$$d_\alpha = d\,|_{X_\alpha \times X_\alpha}$$

for each $\alpha \in \mathfrak{A}$. Then

(a) *for each $\alpha \in \mathfrak{A}$, (X_α, d_α) is a metric space,*

(b) *for any $\alpha, \beta \in \mathfrak{A}$ with $\alpha \neq \beta$,*

$$d(x, y) = +\infty$$

for any $x \in X_\alpha$ and $y \in X_\beta$,

(c) *(X, d) is a generalized complete metric space if and only if, for each $\alpha \in \mathfrak{A}$, (X_α, d_α) is a complete metric space.*

PROOF. Parts (a) and (b) are clear.

(c) Suppose that (X, d) is a generalized complete metric space and, for any $\alpha \in \mathfrak{A}$, let $\{x_n\}$ be a d_α-Cauchy sequence in X_α. Then $\{x_n\}$ is a d-Cauchy sequence in X and, therefore, is d-convergent to $x \in X$. Since clearly limits of sequences are unique and since $d(x_n, x) < +\infty$ for sufficiently large n, $x \in X_\alpha$ and the sequence $\{x_n\}$ is d_α-convergent to x.

Conversely, suppose that, for each $\alpha \in \mathfrak{A}$, (X_α, d_α) is a complete metric space and let $\{x_n\}$ be a d-Cauchy sequence in X. There exists a positive integer N such that $d(x_n, x_m) < +\infty$ for $m, n \geqq N$. Hence there exists an $\alpha \in \mathfrak{A}$ such that $x_n \in X_\alpha$ for $n \geqq N$ and the sequence $\{x_n \mid n \geqq N\}$ is d_α-Cauchy in X_α. Thus $\{x_n \mid n \geqq N\}$ is d_α-convergent to $x \in X_\alpha$. Clearly $\{x_n\}$ is d-convergent to x in X.

3. Fixed points theorem.

3.1. THEOREM. *Let (X, d) be a generalized metric space,*

$$X = \bigcup \{X_\alpha \mid \alpha \in \mathfrak{A}\}$$

the canonical decomposition and let

$$T\colon X \to X$$

be a mapping such that

$$(*) \qquad d(T(x), T(y)) < +\infty$$

whenever $x, y \in X$ and $d(x, y) < +\infty$. Then T has a fixed point if and only if

$$T_\alpha = T|_{X_\alpha}: X_\alpha \to X_\alpha$$

has a fixed point for some $\alpha \in \mathfrak{A}$.

Since the proof is clear, we merely remark that the condition (*) is necessary for T_α to be a mapping from X_α *into* X_α. Furthermore, a "local" version of Theorem 3.1 can be obtained as follows.

3.2. THEOREM. *Let* (X, d) *be a generalized metric space,*

$$X = \bigcup \{X_\alpha \mid \alpha \in \mathfrak{A}\}$$

the canonical decomposition and let

$$T: X \to X$$

be a mapping. If there exists a constant $r > 0$ *such that*

$$(^{**}) \qquad d(T(x), T(y)) < r$$

whenever $d(x, y) < r$, *then* T *has a fixed point if and only if for some subset* $Y \subset X$ *of diameter (defined in the usual manner)* $2r$, *the restriction*

$$T|_Y: Y \to Y$$

has a fixed point.

4. Applications. As illustrative applications, we shall prove a theorem which easily implies the main theorem of [2].

THEOREM. *Let* (X, d) *be a generalized complete metric space,*

$$X = \bigcup \{X_\alpha \mid \alpha \in \mathfrak{A}\}$$

the canonical decomposition and let

$$T: X \to X$$

be a contraction, i.e., there is a constant q *such that* $0 \leqq q < 1$ *and that*

$$d(T(x), T(y)) \leqq q d(x, y)$$

for all $x, y \in X$. *If there exists an* $x_0 \in X$ *such that* $d(x_0, T(x_0)) < +\infty$, *then for some* $\alpha \in \mathfrak{A}$, *the restriction*

$$T_\alpha = T|_{X_\alpha}: X_\alpha \to X_\alpha$$

is a contraction.

PROOF. Since $d(x_0, T(x_0)) < +\infty$, both x_0 and $T(x_0)$ belong to the same X_{α_0}, for some $\alpha_0 \in \mathfrak{A}$. Since T is contractive, $T(X_{\alpha_0}) \subset X_{\alpha_0}$ and the restriction $T_{\alpha_0} = T|_{X_{\alpha_0}}$ is a contraction from X_{α_0} into X_{α_0}.

COROLLARY. *Assuming the same hypotheses as in the theorem, let* $x \in X$ *and consider the "sequence of successive approximations with initial element* x,"

$$x,\ Tx,\ T^2x,\ \cdots,\ T^lx,\ \cdots.$$

Then the following alternative holds: either

(A) *for every integer* $l = 0, 1, 2, \cdots$, *one has*

$$d(T^lx,\ T^{l+1}x) = +\infty$$

or

(B) *the sequence* $x, Tx, T^2x, \cdots, T^lx, \cdots$, *is d-convergent to a fixed point of* T.

PROOF. If (A) does not hold, then for some l, $d(T^lx, T^{l+1}x) < +\infty$. Letting $x_0 = T^lx$, the theorem shows that

$$T_\alpha = T\,|_{X_\alpha} : X_\alpha \to X_\alpha$$

is a contraction, where X_α is the complete metric space containing x_0. By the fixed point theorem of Banach for contractions on complete metric spaces, the sequence $T^lx, T^{l+1}x, \cdots$ is d_α-convergent to a fixed point p of T_α. This implies that (B) holds.

REFERENCES

1. S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux equations integrales* (Doctoral Thesis, University of Lwow), Fund. Math. **3** (1922), 133–181.

2. J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

3. W. A. J. Luxemburg, *On the convergence of successive approximations in the theory of ordinary differential equations*. II, Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 540–546.

4. B. Margolis, *On some fixed points theorems in generalized complete metric spaces*, Bull. Amer. Math. Soc. **74** (1968), 275–282.

KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66502