TWO TYPES OF LOCALLY COMPACT RINGS

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Here we shall present structure theorems for two types of commutative locally compact rings with identity. The first is for rings satisfying a rather stringent topological condition, namely, that there exist an invertible, topologically nilpotent element. An analysis of such rings requires basic theorems of commutative algebra and, in particular, a decomposition theorem for total quotient rings of one-dimensional Macaulay rings. One consequence of the structure theorem is the determination of necessary and sufficient conditions for a locally compact ring with identity to be the topological direct product of topological algebras over indiscrete locally compact fields.

The second is a theorem classifying all compatible metrizable locally compact topologies on a ring satisfying very stringent algebraic conditions, namely, that the ring be a special principal ideal ring in the sense of Zariski and Samuel [6, p. 245] of either zero or prime characteristic. For this investigation we require a theorem concerning finite-dimensional, locally compact, metrizable vector spaces over discrete fields, which shows that, in a certain sense, such spaces are not too remote from finite-dimensional vector spaces over indiscrete locally compact fields.

1. Commutative locally compact rings having an invertible, topologically nilpotent element. We recall that a *local ring* is a commutative ring with identity that has only one maximal ideal, and that the *natural topology* of a local noetherian ring is obtained by declaring the powers of its maximal ideal a fundamental system of neighborhoods of zero. Moreover, a local noetherian ring is compact for its natural topology if and only if it is complete and its residue field is finite. If A is a compact ring that algebraically is a local noetherian ring, then the topology of A is its natural topology [4, Theorem 4]. We recall also that a one-dimensional local noetherian ring is a Macaulay ring if and only if its maximal ideal is not an associated prime ideal of the zero ideal [7, p. 397].

Let B be a one-dimensional Macaulay ring topologized with its natural topology, and let $\mathfrak{m}$ be its maximal ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ the (isolated) prime ideals of the zero ideal. The complement of $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$ is the set of cancellable elements of B. Let A be the total quotient

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ring of B, topologized by declaring the neighborhoods of zero in B to be a fundamental system of neighborhoods of zero in A; this topology we call the *B-topology* on A. To show that A is a topological ring, it suffices to show that $x \rightarrow b^{-1}x$ is continuous at zero for any cancellable $b \in B$; this is accomplished by observing that $\mathfrak{m}$ is the only prime ideal of Bb. Any element of $\mathfrak{m}$ not belonging to $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$ is an invertible, topologically nilpotent element of A.

We shall say that a local noetherian ring is *aligned* if the prime ideals, ordered by inclusion, form a chain. Thus a one-dimensional aligned local noetherian ring has precisely two proper prime ideals, one contained in the other. The decomposition theorem needed is the following:

THEOREM 1. *If A is the total quotient ring of a one-dimensional Macaulay ring B and if A is topologized by the B-topology, then A is the topological direct product of ideals $A_1, \cdots, A_n$, where each A_k is the total quotient ring of a one-dimensional aligned Macaulay ring B_k and is topologized by the B_k-topology.*

A *semilocal ring* is a commutative ring with identity that has only finitely many maximal ideals. A *Cohen algebra* is a local algebra over a field whose maximal ideal has codimension one. Our structure theorem for commutative locally compact rings having an invertible, topologically nilpotent element is the following:

THEOREM 2. *Let A be a commutative locally compact ring with identity. The following statements are equivalent:*

1°. *A contains an invertible element that is topologically nilpotent.*

2°. *A is semilocal, and none of its maximal ideals is open.*

3°. *A is the topological direct product of a sequence $(A_k)_{1 \leq k \leq n}$ of ideals where each A_k is either a locally compact finite-dimensional Cohen algebra over the topological field of real or complex numbers or the topological quotient ring of a compact one-dimensional aligned Macaulay ring.*

OUTLINE OF PROOF. We first assume that A is totally disconnected and satisfies 1°. By a lemma of Kaplansky [3, Lemma 5], A contains a compact open subring B that contains the identity element of A. By Kaplansky's characterization of compact semisimple rings [2, Theorem 16], the existence of an invertible, topologically nilpotent element implies that the radical R of B is open, so B/R is the cartesian product of finitely many finite fields. Raising idempotents from B/R to B and then using them to decompose A, we conclude that A is the topological direct product of ideals $A_1, \cdots, A_m$ where each A_i contains a compact, open, local subring B_i. Once again the hypothesis that A_i has an invertible topologically nilpotent element implies that all the powers of the radical of B_i are open, so by a theorem of Kaplansky [2, Theorem 20], B_i is a local noetherian ring. It is easy to see, in fact, that B_i is a one-dimensional Macaulay ring and that A_i is the total quotient ring of B_i equipped with the B_i-topology. An application of Theorem 1 to each A_i shows that 3° holds and, in particular, that the radical of A is nilpotent. The general case is now established by use of the Pontryagin-van Kampen theorem on commutative, locally compact, connected groups, the nilpotence of the radical in the totally disconnected case, and the fact, proved by using standard theorems concerning finite-dimensional topological vector spaces over locally compact fields, that a locally compact local ring whose maximal ideal is nilpotent but not open is either connected or totally disconnected [5, Lemma 7].

THEOREM 3. *Let A be a commutative locally compact ring with identity. The following statements are equivalent*:

1°. *A contains an invertible element that is topologically nilpotent, and the additive order of each element of A is either infinite or a square-free integer.*

2°. *A is semilocal, none of its maximal ideals is open, and the additive order of each element of A is either infinite or a square-free integer.*

3°. *A is the topological direct product of topological algebras over indiscrete locally compact fields.*

4°. *A is the topological direct product of finitely many finite-dimensional Cohen algebras over indiscrete locally compact fields.*

OUTLINE OF PROOF. To show that 1° implies 4°, it suffices by Theorem 2 to consider the case where A is the total quotient ring of a one-dimensional aligned compact Macaulay ring B, equipped with the B-topology. Then A is local, and its maximal ideal $\mathfrak{m}$ is closed, not open, and nilpotent. Consequently, A and $A/\mathfrak{m}$ have the same characteristic. It follows easily that A contains an indiscrete topological subfield K that is the quotient field of a principal ideal domain D and that the open D-submodules of K form a fundamental system of neighborhoods of zero in K. We may therefore apply Correl's theorem [1, Theorem 3] to conclude that the completion of K is a locally compact field. A modification of a proof of I. S. Cohen's theorem on complete equicharacteristic local rings [7, pp. 304–306] enables us to replace this field by a locally compact subfield that is canonically mapped onto $A/\mathfrak{m}$ [5, Lemma 5].

COROLLARY. *Let A be a locally compact ring with identity. The following statements are equivalent*:

1°. *The center of A contains an invertible element that is topologically nilpotent, and the additive order of each element of A is either infinite or a square-free integer.*

2°. *A is the topological direct product of finitely many topological algebras over indiscrete locally compact fields.*

2. **Locally compact metrizable special principal ideal rings.** A special principal ideal ring [6, p. 245] is a principal ideal ring that has only one proper prime ideal, and that ideal is nilpotent. If A is a special principal ideal ring whose characteristic is either zero or a prime, then by I. S. Cohen's theorem A is an algebra over a field K that has a basis $1, c, c^2, \ldots, c^{s-1}$, where $c^s = 0$. Suppose that K admits an indiscrete locally compact topology compatible with its field structure, and let $r \in [0, s-1]$. Then Ac^r is the finite-dimensional subspace generated by $c^r, \ldots, c^{s-1}$ and hence admits a unique topology making it a Hausdorff vector space over K. We topologize A by declaring the neighborhoods of zero in Ac^r to be neighborhoods of zero in A. It is not difficult to verify that A, so topologized, is a topological ring (though if $r > 0$, A is a topological algebra over K only if K is given the discrete topology). This topology depends only on the topological field K and the numbers r and s, so we shall call it the (K, r, s)-topology. To show that every compatible locally compact metrizable topology on A is a (K, r, s)-topology, we require the following theorem:

THEOREM 4. *Let E be a totally disconnected, finite-dimensional, locally compact, metrizable vector space [algebra] over a discrete field K, and let $L = \{x \in E$: either $x = 0$ or Kx is indiscrete$\}$. Then L is an open subspace [open ideal] of E, and L is the topological direct sum of subspaces [ideals of E] $E_1, \ldots, E_n$, where for each $i \in [1, n]$, the locally compact group [ring] E_i admits the structure of finite-dimensional topological vector space [algebra] over an indiscrete locally compact field F_i under a scalar multiplication satisfying $\alpha.(\mu x) = \mu(\alpha.x)$ [and also $\alpha.(xy) = (\alpha.x)y$, $\alpha.(yx) = y(\alpha.x)$] for all $\alpha \in F_i$, $\mu \in K$, $x \in E_i$ [and $y \in E$]. If N is any algebraic supplement of L, then N is discrete, and E is the topological direct sum of $E_1, \ldots, E_n, N$.*

Other than elementary facts, the proof depends only on the following three theorems: (1) The Baire Category Theorem (to show that if E is indiscrete, then K is uncountable); (2) There exist no nonzero compact metrizable vector spaces over an uncountable discrete field (a consequence of the theory of characters); (3) The Open Mapping Theorem for separable, metrizable, locally compact groups. An analogue of Theorem 4 also holds for connected, locally compact, metrizable vector spaces and algebras over discrete fields.

THEOREM 5. *Let A be an indiscrete, locally compact, metrizable, special principal ideal ring whose characteristic is either zero or a prime, and let $\mathfrak{m}$ be the maximal ideal of A. The topology of A is then the (K, r, s)-topology, where K is an indiscrete locally compact field that algebraically is a subfield of A and is mapped canonically onto $A/\mathfrak{m}$, where r is the largest integer such that $\mathfrak{m}^r$ is open, and where s is the index of nilpotence of $\mathfrak{m}$.*

OUTLINE OF PROOF. From the existence of a coefficient field, Theorem 4, and the fact that a field cannot admit both a connected and a totally disconnected locally compact topology compatible with its field structure, it follows that $\mathfrak{m}^r$ with its induced topology is a connected or totally disconnected, metrizable, locally compact finite-dimensional algebra over an indiscrete locally compact field F. Using again the special case of I. S. Cohen's theorem for equicharacteristic local rings having a nilpotent maximal ideal, we find a subfield K_0 of A that is algebraically isomorphic to F and acts on $\mathfrak{m}^r$ as F does. Transferring the topology of F to K_0 and applying [5, Lemma 5], we obtained the desired conclusion.

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