RESEARCH ANNOUNCEMENTS

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WALL'S SURGERY OBSTRUCTION GROUPS FOR $Z\times G$, FOR SUITABLE GROUPS G

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Introduction. Let $\mathcal{C}$ be the category whose objects are pairs (G, w), G a group and w a homomorphism of G into Z_2, and whose morphisms are the obvious ones. Every finite Poincaré complex X^n determines functorially an element $(\pi_1(X), w(X))$ of this category; let $w(X)(b)=1$ if b preserves orientation and let $w(X)(b)=-1$ otherwise. There is a sequence of functors L_n, $n\geqq 5$, from $\mathcal{C}$ to the category of abelian groups, with $L_n=L_{n+4}$ all $n\geqq 5$, which plays the role of the range of a surgery obstruction. More precisely, let X^n be a compact smooth manifold and let v be its stable normal bundle. (Actually, one only needs a finite Poincaré complex with a given vector bundle; see [3] and [4].) Let $\Omega_n(X, v)$ be the cobordism classes of triples (M, ϕ, F), M a compact smooth manifold, ϕ: $(M, \partial M)\to(X, \partial X)$ a map of degree one which induces a homotopy equivalence of boundaries, and F a stable framing of $t(M)\oplus\phi^*v$, where $tM=$tangent bundle of M. Then for $n\geqq 5$, there is a map θ: $\Omega_n(X, v)\to L_n(\pi_1(X), w(X))$ such that $\theta[M, \phi, F]=0$ if and only if this class $[M, \phi, F]$ contains (N, ψ, G) with ψ a homotopy equivalence. For n even, the functors L_n and this map are defined by Wall in [3]. For n odd, they are defined by Wall in [4], but with "homotopy equivalence" replaced by "simple homotopy equivalence." However, one can slightly alter the procedures of [4] to define L_n and θ with the properties just mentioned.

The groups $L_n(\pi_1X, wX)$ are not too large in the sense that every one of their elements is the obstruction to some surgery problem with boundary. In fact, we have the following result, due essentially to Wall (see [3, p. 274] and [4, §5, 6]).

THEOREM 0.1. *Let X^{m-1}, $m\geqq 6$, be a compact connected smooth manifold. Let v be the stable normal bundle of X. Let γ be a given element of $L_m(\pi_1X, wX)=L_m(\pi_1(X\times I), w(X\times I))$. Let ϕ_1 be a homotopy equivalence of M^{m-1} and X, M a compact smooth manifold, which induces a homotopy equivalence of boundaries. Let F_1 be a stable framing of*

$tM\oplus\phi^*v$. Then there is a map of smooth manifold triads (with corners),

$$\phi\colon (W, \partial_-W, \partial_+W) \to (X\times I, X\times 0\cup \partial X\times I, X\times 1),$$

and a stable framing F of $tW\oplus\phi^*(v\times I)$ such that

(1) *$\partial_-W = M\times 0\cup\partial M\times I$ and $\phi(x, t) = (\phi_1 x, t)$ if $x\in\partial M$ or $x\in M$ and $t=0$;*
(2) *$\phi|\partial_+W\colon \partial_+W\to X\times 1$ is a homotopy equivalence;*
(3) *F extends F_1 $(tW\oplus\phi^*(v\times I)|M = tM\oplus\phi_1{}^*v\oplus\epsilon')$; and*
(4) *$\theta[W\ \phi, F] = \gamma$.*

Hence it seems interesting to try to compute the groups $L_n(G, w)$. The main result of this note, Theorem 1.1, is the computation of $L_n(G\times Z, w_1)$ in terms of $L_n(G, w)$ and $L_{n-1}(G, w)$, provided that the abelian group $C(G, id)=0$. Here Z denotes the integers and w_1 is the composite $G\times Z\to G\xrightarrow{w} Z_2$, while if α is any automorphism of G, $C(G,\alpha)$ is defined in [1] as follows: Let $\mathcal{C}(G, \alpha)$ be the abelian category whose objects are pairs (P, f), P a finitely generated projective $Z(G)$-module and f an α-semilinear nilpotent endomorphism of P; and whose morphisms are $Z(G)$-module maps which preserve the nilpotent endomorphisms. Let $\mathcal{C}_1$ be the set of isomorphism classes of objects of $\mathcal{C}(G, \alpha)$. Then $C(G, \alpha)$ is the range of the additive map of $\mathcal{C}_1$ to an abelian group which vanish on free modules with the zero endomorphism and which is universal among such maps. If G is free abelian and finitely generated, $C(G, \alpha)=0$ (because $Z(G)$ is regular and $\tilde{K}_0(Z(G))=0$).

1. Statement of results.

THEOREM 1.1. *Let K^{n-2}, $n\geq 7$, be a compact connected smooth manifold with fundamental group G, and let $w\colon G\to Z_2$ be its orientation map. Assume that $C(G, id)=0$ and that for any choice of basepoint, $C(\pi_1(\partial K), id)=0$. Let w_1 be the composite of w and the natural projection of $G\times Z$ onto G. Then there is a split exact sequence*

$$0\to \ker\alpha(K)\to L_n(G\times Z, w_1)\xrightarrow{\alpha(K)} L_{n-1}(G, w)\to 0$$

and an isomorphism $\beta(K)\colon \ker\alpha(K)\to L_n(G, w)$. Let v be the stable normal bundle of $K\times I$. Then there is a splitting $i(K)$ of the exact sequence above such that the following diagram commutes (ϵ = the trivial line bundle over S^1):

$$\begin{array}{ccc} \Omega^*_{n-1}(K\times I, v) & \xrightarrow{\times S'} & \Omega^*_n(K\times I\times S^1, v\times\epsilon) \\ \downarrow\theta & & \downarrow\theta \\ L_{n-1}(G, w) & \xrightarrow{i(K)} & L_n(G\times Z, w_1). \end{array}$$

NOTE. By $\Omega_n^*(X, v)$ we denote those classes of triples which contain an element of the form (W, ϕ, F) with $\partial_- W = X$ and $\phi(x) = (x, 0)$ for x in $\partial_- W$.

In particular, since the groups $L_n(H)$ are well known if H is the trivial group, this theorem allows us to compute $L_n(Z^k)$ for all k, where Z^k is the free abelian group on k generators. This computation has been found independently by C. T. C. Wall (private communication).

THEOREM 1.2. *Let ϕ: $(M, \partial M) \to (X, \partial X)$ be a map of degree one of connected compact smooth manifolds which induces a homotopy equivalence of boundaries. Assume $n = \dim X = \dim M \geqq 5$ if ∂X is empty and $\geqq 6$ if not. Let v be the stable normal bundle of X, and let F be a stable framing of $tM \oplus \phi^* v$. Let $F \times S^1$ be the corresponding stable framing of $t(M \times S^1) \oplus (\phi \times S^1)^*(v \times S^1) = (tM \oplus \phi^* v) \times \epsilon$. Assume that $C(G, id) = 0$ and that for any basepoint $C(\pi_1(\partial X), id) = 0$, where $G = \pi_1 X$. Then the following are equivalent*:

(1) *(M, ϕ, F) is cobordant to (N, ϕ_1, G) with ϕ_1 a homotopy equivalence*; *and*

(2) *$(M \times S^1, \phi \times S^1, F \times S^1)$ is cobordant to (P, ψ, E) with ψ a homotopy equivalence.*

2. **Outline of proofs.** The first step is to recast a portion of the main result of Farrell's thesis [1]. Suppose that K^n, $n \geqq 6$, is a compact connected smooth manifold and that f: $K \to S^1$ is a map with regular value $*$, the basepoint of S^1. Suppose also that $*$ is a regular value of $f | \partial K$, and let $N = f^{-1}(*)$. Assume that the map of fundamental groups induced by f is an epimorphism with kernel G, and assume that the covering space of K associated to G is dominated by a finite complex. Assume that f also has the following property: $(\partial K)_{\partial N}$, the manifold obtained by splitting ∂K along ∂N, is an h-cobordism. Then one can still define $c(f)$ in $C(G, \alpha)$ as in [1], and as in [1] one can prove the following:

THEOREM 2.1. *Under the above hypotheses, the following are equivalent*:

(1) *f is homotopic relative ∂K to a map g such that $*$ is a regular value of g and such that K_M, the manifold obtained by splitting K along M, is an h-cobordism, $M = g^{-1}(*)$* (i.e. *g_*: $\pi_i(K, M) \to \pi_i(S^1, *)$ is an isomorphism all i and M is connected*); *and* (2) $c(f) = 0$.

Note. In [1] Farrell defines another obstruction $\tau(f)$ in a quotient group of $\mathrm{Wh}(G)$. He shows that f is homotopic rel the boundary to a differentiable fibration if and only if $\tau(f)$ and $c(f)$ both vanish.

Now suppose that g: $X^n \to S^1$, $n \geqq 6$ if ∂X is empty and $n \geqq 7$ otherwise, is a differentiable fibration of the compact connected manifold X whose restriction to the boundary is also a fibration. Assume that the map induced by g on fundamental groups is an epimorphism with kernel G, and that $C(G, \alpha)=0$. Here α is the automorphism of G determined by conjugation with an element of $\pi_1(X)$ that is carried onto the standard generator of $\pi_1 S^1$ by g_*, as in [1]. Assume also that the restriction of g to each component of ∂K induces an epimorphism of fundamental groups, and that if H is any one of the kernels of these induced maps and α_H the corresponding automorphism of H, then $C(H, \alpha_H)=0$. Let v be the stable normal bundle of X. Let (M, ϕ, F) represent an element of $\Omega_n(X, v)$. After a homotopy of ϕ as a map of the pair $(M, \partial M)$ to the pair $(X, \partial X)$, we can assume that $*$ is a regular value of $g\phi$ and of $g\phi|\partial M$. By Theorem 2.1 we can also suppose that $\phi|\partial N\colon \partial N \rightarrow \partial L$ is a homotopy equivalence, where L is the fibre of g and $N=\phi^{-1}(L)$. Let $\psi=\phi|N\colon N\rightarrow L$. $F|N$ is a stable framing of $\psi^*(v|L)\oplus tN$ and $v|L$ is the stable normal bundle of L. Define

$$\alpha_g(M, \phi, F) = \theta(N, \psi, F \mid N) \in L_{n-1}(G, w(X)\mid L).$$

PROPOSITION 2.2. *$\alpha_g(M, \phi, F)$ is well defined and depends only upon the cobordism class of (M, ϕ, F). If ϕ is a homotopy equivalence, then $\alpha_g(M, \phi, F)=0$.*

The proof of this proposition is a fairly straightforward, if tedious, application of Theorem 2.1 and the basic properties of θ and of the L_n. Theorem 1.2 is an easy consequence of Proposition 2.2 for the special case where $X=L\times S^1$ and g is the projection on the second factor.

Now suppose that $\alpha_g(M, \phi, F)=0$. Then (M, ϕ, F) is cobordant to (M_1, ϕ_1, F_1), where $*$ is a regular value of $g\circ\phi_1$ and where, putting $N_1=\phi_1^{-1}(L)$, the restriction of ϕ_1 to N_1 is a homotopy equivalence of N_1 and L. Let X_L be the manifold obtained by splitting X along L. Let v_L be the bundle induced from v by the natural map of X onto X_L. Let P be the manifold obtained by splitting M_1 along N_1. Then ϕ_1 induces a map ϕ_L of P into X_L which restricts to a homotopy equivalence of boundaries. Moreover $tP\oplus\phi_L{*}v_L$ is induced from $tM_1\oplus\phi_1^*v$ by the natural map of P onto M_1; hence F_1 induces a stable framing F_L of this (stable) vector bundle. Define

$$\beta_g(M, \phi, F) = \theta(P, \phi_L, F_L) \in L_n(G, w(X_L)).$$

PROPOSITION 2.3. *β_g is well defined on triples (M, ϕ, F) with $\alpha_g(M, \phi, F)=0$ and depends only upon the cobordism class of such a triple in $\Omega_n(X, v)$. If ϕ is a homotopy equivalence, then $\beta_g(M, \phi, F)=0$.*

Thus we have, under the hypotheses stated in the paragraph following Theorem 2.1, two necessary conditions that (M, ϕ, F) be cobordant to (N, ψ, G), with ψ a homotopy equivalence. These conditions are also sufficient.

THEOREM 2.4. *The following are equivalent*:

(1) $\alpha_g(M, \phi, F)=0$ *and* $\beta_g(M, \phi, F)=0$; *and*

(2) $\theta(M, \phi, F)=0$.

In fact, it follows from the definitions of α_g and β_g that if (1) holds, we can, after a cobordism (of triples), assume that if $\phi^{-1}(L)=N$, N is a proper submanifold of M and ϕ induces homotopy equivalences of N with L and of M_N with X_L. But in this case, ϕ is itself a homotopy equivalence; this follows by taking the universal cover of S^1 and analyzing the covering spaces over M and X induced by g and $g\phi$ respectively.

Now assume the hypotheses of Theorem 1.1 and set $X=L\times S^1$. Let g be the canonical projection of X onto S^1. Then using Proposition 2.2 and Theorem 0.1, the correspondence $\theta[M,\phi, F]\to\alpha_g[M,\phi, F]$ defines a homomorphism $\alpha(L)\colon L_n(G\times Z, w_1)\to L_{n-1}(G, w)$. If $i(L)$ is defined by taking the product with S^1 of representatives in $\Omega_{n-1}(L\times I, v)$ of elements in $L_{n-1}(G, w)$, it is easy to see that $\alpha(L)\circ i(L)=id$. By applying Proposition 2.3 and Theorems 2.4 and 0.1 one can construct similarly an isomorphism $\beta(L)\colon \ker\alpha(L)\to L_n(G, w)$

REFERENCES

1. F. T. Farrell, *The obstruction to fibering a manifold over* S^1, thesis, Yale University, 1967.

2. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. **77** (1963), 504–537.

3. C. T. C. Wall, *Surgery of non-simply-connected manifolds*, Ann. of Math. **82** (1966), 217–276.

4. ———, *Surgery of non-simply-connected manifolds*, notes, University of Liverpool, 1967.

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