THE WEAKLY COMPLEX BORDISM OF LIE GROUPS

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1. **Preliminaries.** Let $\mathcal{K}$ be the class of compact 1 connected semisimple Lie groups; $\mathcal{K}'\subset\mathcal{K}$ is the following set of groups, $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$, G_2, F_4, E_6, E_7, E_8, $U_*(X)$ the weakly complex bordism of X [1] and Λ the ring $U_*(pt)=Z[Y_1, Y_2, \cdots]$. Λ is the weakly complex bordism ring defined by Milnor. The generators Y_i are weakly complex manifolds of dim $2i$. The bordism class of a weakly complex manifold M^{2n} is determined by its Milnor numbers [2] $s_\omega[M^{2n}]$ for ω ranging over all partitions of n. In particular, the generators Y_i can be chosen so that $s_i(Y_i)=1$ unless $i=p^k-1$ for some prime p and in this case $s_i(Y_i)=p$; moreover, we assume generators Y_i chosen so that its Todd genera are 1.

It is possible and convenient to introduce bordism theories with other coefficient rings than Λ. If Γ is such a ring, $U_*(\ ,\Gamma)$ will denote the resulting theory. Briefly here are some examples: $\Lambda_p=Z_p[Y_1, Y_2, \cdots]$, $\Lambda[1/Y_{p-1}]=\text{direct lim } 1/Y^n_{p-1}\Lambda$ and $\Lambda_p[1/Y_{p-1}]=\text{direct lim } 1/Y^n_{p-1}\Lambda_p$.[1] Let $M=\{M_n\}$ denote the stable object of Milnor [1] and $Z_p=S^1U_pE^2$ the space obtained by attaching E^2 to S^1 via a map of degree p. $M^{Z_p}_{n+2}$ denotes the space of base point preserving maps from Z_p to M_{n+2}. Then $U_k(X,\Lambda_p)=\text{direct lim } \Pi_{n+k}(X^+\wedge M^{z_p}_{n+2})$ X^+ is the disjoint union of X and a point $x_0\cdot U_*(X,\Lambda_p)$ is the resulting theory. $U_*(X,\Lambda[1/Y_{p-1}])=U_*(X)\otimes_\Lambda\Lambda[1/Y_{p-1}]$ and $U_*(X,\Lambda_p[1/Y_{p-1}])=U_*(X,\Lambda_p)\otimes_{\Lambda_p}\Lambda_p[1/Y_{p-1}]$.

To $K\subset\mathcal{K}$ there is associated a "generating variety" K_s introduced by Bott [4]. Essentially K_s is the homogeneous space K/K^s where K^s is the centralizer of a 1-dimensional torus $S^1\subset K$. The dimension of the center of K^s is 1. The commutator map

$$S^1\times K_s\xrightarrow{[\ ]}K$$

defined by $[t,[k]]=tkt^{-1}k^{-1}$ for $[k]\in K_s$, $t\in S^1\subset K$ is of particular importance.

2. **Statement of results.** Define $\Lambda(K)=\Lambda$ if $H^*(K)$ has no torsion, $=\Lambda[1/Y_1]$ if $H^*(K)$ has only 2 torsion, $=\Lambda[1/Y_1, 1/Y_2]$ if $H^*(K)$ has only 2, 3 torsion, $=\Lambda[1/Y_1, 1/Y_2, 1/Y_4]$ if $H^*(K)$ has 2, 3 and 5 torsion.

[1] E.g., $\Lambda[1/Y_{p-1}]$ is the ring obtained from Λ by making Y_{p-1} a unit.

THEOREM 1. *If* $K=\mathrm{Spin}\,(n)$, $\mathrm{Sp}\,(n)$, $\mathrm{SU}\,(n)$ *or* G_2, $\mathrm{Im}\,[\ \ ]_*$ *generates* $U_*(K,\Lambda(K))$ *and* $E_0U_*(K,\Lambda(K))$ *is an exterior algebra on rank* K *generators for some filtration of* $U_*(K,\Lambda(K))$.

THEOREM 2. *If* $K\subset\mathcal{K}'$ *then* $\mathrm{Im}\,[\ \ ]_{p*}$ *generates* $U_*(K,\Lambda_p[1/Y_{p-1}])$ *and* $E_0U_*(K,\Lambda_p[1/Y_{p-1}])$ *is an exterior algebra on rank* K *generators (except possibly for* $(E_7,2)$, $(E_8,2)$, $(E_8,3)$). p *is a prime.*

COROLLARY 3. *If* $K\subset\mathcal{K}'$, $\mathrm{Im}\,[\ \ ]_*$ *generates (algebraically)* $U_*(K,\Lambda(K))$ *and* $U_*(K,\Lambda(K))$ *is a torsion free abelian group (except possibly for* E_7 *and* E_8).

COROLLARY 4 (HODGKIN). *For* K *as in Theorem* 1, $K^*(K)$ *is an exterior algebra on rank* K *generators.*

THEOREM 5. *For* $n\geqq 7$ $U_*(\mathrm{Spin}\,(n))$ *has* 2 *torsion and* Y_1 *torsion*

THEOREM 6. *For any* i, *the* Y_i *torsion subgroup of* $U_*(\mathrm{Spin}\,(n))$ *is contained in the* Y_1 *torsion subgroup of* $U_*(\mathrm{Spin}\,(n))$.

3. **Outline of techniques.** The most significant fact about the $K\subset\mathcal{K}$ is that the homology of $G=\Omega K$ is all even dimensional and generated by weakly complex manifolds [4], [5]. The method we have chosen to exploit this fact is the following: The Milnor construction of the classifying space K of ΩK leads to a spectral sequence converging to $U_*(K)$ [6]. The E^2 term in this case is $\mathrm{Tor}^{U_*(G)}(\Lambda,\Lambda)$ because $U_*(G)$ is Λ free. (This follows from the fact.) Introducing Γ coefficients, there results a spectral sequence $\mathrm{Tor}^{U_*(G,\Gamma)}(\Gamma,\Gamma)\Rightarrow U_*(K,\Gamma)$. The ring $U_*(G,\Gamma)$ is determined for various Γ. $\mathrm{Tor}^{U_*(G,\Gamma)}(\Gamma,\Gamma)$ is shown to be an exterior algebra on rank K generators and consequently the spectral sequence collapses. The generators lie in $E^\infty_{1,*}$. This implies that $\mathrm{Im}\,[\ \ ]_*$ generates $U_*(K,\Gamma)$.

There is a procedure for passing from the homology ring $H_*(G)$ to the ring $U_*(G)$. It is this: Let $\mu\colon U_*(G)\to H_*(G)$ be the natural transformation defined by $\mu[M,f]=f_*(\sigma_M)$ see [1]. $H_*(G)$ is $Z[w_1,w_2,\cdots,w_n]/I$ as an algebra where the w_i are even dimensional and I is an ideal $(f_1(w),f_2(w),\cdots,f_k(w))$. f_i is a homogeneous polynomial in the w_i. Let $\Gamma\in U_*(G)$ be such that $\mu(\Gamma_i)=w_i$ and suppose each Γ_i augments to zero under $U_*(G)\to U_*(pt)$. Then $U_*(G)=\Lambda[\Gamma_1,\Gamma_2,\cdots,\Gamma_k]/J$ as an algebra where J is the ideal generated by $(*)$ $g_i(\Gamma)=f_i(\Gamma)+\sum_j V_{ij}m_{ij}(\Gamma)$, $i=1\cdots k$. Here $m_{ij}(\Gamma)$ is a monomial in the Γ_j's of total dimension strictly less than that of $f_i(\Gamma)$ and $V_{ij}\in\Lambda$. Using the characteristic classes s_ω [2], one can define characteristic numbers $s_\omega(\alpha)$ for $\alpha\in U_*(G)$ and $\alpha=0$ iff all characteristic numbers $s_\omega(\alpha)$ are zero. Since $g_i(\Gamma)=0$ we have $s_\omega(g_i(\Gamma))=0$. Expressing this via (*), (**) $s_\omega(f_i(\Gamma))+\sum_j s_\omega(V_{ij}m_{ij}(\Gamma))=0$. Expanding this further gives a sequence of linear equations involving the characteristic numbers $s_\omega[V_{ij}]$ and known quantities. One solves for the $s_\omega[V_{ij}]$'s which completely determines $V_{ij}\in\Lambda$.

The data necessary to solve the equation (**) is: (1) A choice of weakly complex manifolds M_i and maps $f_i: M_i\rightarrow G$ such that $\{f_{i*}(\sigma_{M_i})\}$ generate the ring $H_*(G)$, (2), the ring $H^*(M_i)$, (3), the Milnor characteristic classes of M_i and the ring homomorphisms f_j^*. Part of this is supplied in [4] and [5]; the remainder by the author.

Having obtained the ring $U_*(G, \Gamma)$ one uses homological algebra and determines the algebra $\mathrm{Tor}^{U_*(G,\Gamma)}(\Gamma, \Gamma)$ from which the theorems follow.

Bibliography

1. P. Conner and E. Floyd, *Torsion in* SU *bordism*, Mem. Amer. Math. Soc. **60** (1966), 72 pp.

2. J. Milnor, *Lectures on characteristic classes*, Mimeographed notes, Princeton Univ., Princeton, N. J., 1958.

3. S. Araki and H. Toda, *Multiplicative structures in* mod *q cohomology theories* 1, Osaka Math J. **2** (1965), 71–115.

4. R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.

5. ———, *Applications of the theory of Morse to symmetric spaces*, Amer J. Math. **80** (1958), 965–1029.

6. J. Milnor, *Construction of universal bundles*. II, Ann. of Math. **63** (1956), 430–436.

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