# AN INEQUALITY FOR THE EIGENVALUES OF A CLASS
## OF SELF-ADJOINT OPERATORS

BY WILLIAM STENGER[1]

**1. Introduction.** The theory of intermediate problems introduced by A. Weinstein in 1935 has not only been of great importance in its application to the numerical computation of lower bounds for eigenvalues, but has also been useful in the investigation of many general questions of a purely theoretical nature. Among these results, let us mention Weinstein's new maximum-minimum theory [1], [2], Stenger's analogous treatment of Poincaré's inequalities [3], and a remarkable inequality of Aronszajn [5, p. 476]. In this paper we give a seemingly new inequality, which is in a sense complementary to Aronszajn's inequality.

**2. The new inequality.** Let $A$ be a self-adjoint linear operator, defined on a dense subspace $D$ of a complex Hilbert space $H$ having the scalar product $(u, v)$. Let $H'$ be a closed subspace of $H$, and let $H'' = H \ominus H'$. Denote by $P'$ and $P''$ the projections onto $H'$ and $H''$, respectively. We assume that the lower parts of the spectra of (i) $A$, (ii) $P'AP'$, and (iii) $P''AP''$ consist of isolated eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_1' \leq \lambda_2' \leq \cdots$, and $\lambda_1'' \leq \lambda_2'' \leq \cdots$, each having finite multiplicity. Obviously, these hypotheses on (i), (ii) and (iii) are satisfied for a nonpositive self-adjoint compact $A$. Moreover, many important Schroedinger-type operators satisfy the hypothesis for (i). It has been recently shown [4] that, under certain conditions, the hypotheses are also true for (ii) and (iii). Let $u_1, u_2, \cdots, u_1', u_2', \cdots$, and $u_1'', u_2'', \cdots$ be the corresponding orthonormal sequences of eigenvectors.

THEOREM. *The eigenvalues of the operators $A$, $P'AP'$, and $P''AP''$ satisfy the inequality*

(1) $$\lambda_1 + \lambda_{i+j} \leq \lambda_i' + \lambda_j'', \quad i, j = 1, 2, \cdots.$$

PROOF. Let $R(u)$ denote the Rayleigh quotient $(Au, u)/(u, u)$. For any $u \in H$, $(u, u) = 1$, we can write $u = \sigma v + \tau w$, where $v \in H'$, $w \in H''$, $(v, v) = (w, w) = 1$, and $|\sigma|^2 + |\tau|^2 = 1$. If $\tau = 0$, we have $u = v$, $R(u) = R(v)$, and $\lambda_1 \leq R(w)$ for any $w$. Therefore, we have

(2) $$R(u) + \lambda_1 \leq R(v) + R(w).$$

We prove now that (2) is true for $\tau \neq 0$. In fact, we have by direct computation

$$
\begin{aligned}
(3) \quad 0 \leq \frac{R(u) - \lambda_1}{|\tau|^2} = &\left|\frac{\sigma}{\tau}\right|^2 [R(v) - \lambda_1] + \left(\frac{\sigma}{\tau}\right)([A - \lambda_1 I]v, w) \\
&+ \left(\frac{\bar{\sigma}}{\tau}\right)\overline{([A - \lambda_1 I]v, w)} + [R(w) - \lambda_1].
\end{aligned}
$$

Now we apply the following lemma, which was used in the proof of Aronszajn's inequality, see [6, p. 76].

LEMMA. *Let $\alpha$, $\gamma$ be any nonnegative constants and let $\beta$ be any complex number such that*

$$
f(z) = \alpha z \bar{z} + \beta z + \overline{\beta z} + \gamma \geq 0
$$

*for all complex $z$. Then we have*

$$
(4) \qquad\qquad f(z) \leq (1 + z\bar{z})(\alpha + \gamma).
$$

Putting $z = \sigma/\tau$, $\alpha = R(v) - \lambda_1$, $\beta = ([A - \lambda_1 I]v, w)$ and $\gamma = R(w) - \lambda_1$, we apply the inequality (4) in (3) and obtain again the inequality (2). Therefore (2) is true for every $u \in H$. Now let $U_i' = \mathrm{sp}\{u_1', u_2', \cdots, u_i'\}$ and $U_j'' = \mathrm{sp}\{u_1'', u_2'', \cdots, u_j''\}$ for $i, j = 1, 2, \cdots$. Using the inequality (2) and a minimum-maximum principle, see [3], we get

$$
\lambda_{i+j} \leq \max_{u \in U_i' \oplus U_j''} R(u) \leq \max_{v \in U_i'} R(v) + \max_{w \in U_j''} R(w) - \lambda_1 = \lambda_i' + \lambda_j'' - \lambda_1
$$

which proves the theorem.

The inequality (1) is a counterpart to the inequality of Aronszajn

$$
(5) \qquad\qquad \lambda_i' + \lambda_j'' \leq \lambda_{i+j-1}, \quad i, j = 1, 2, \cdots,
$$

which is true for nonpositive compact operators. In another paper we shall give the necessary and sufficient conditions, on the spaces $H'$ and $H''$, in order that equality hold in the inequality (5).

3. **Application.** We now give an application of our inequality (1) to the uniform estimation of the error in Weinstein's lower bounds for eigenvalues. Let $p \in D$ such that $(p, p) = 1$ and $(Ap, p) \neq 0$, and let $H' = \mathrm{sp}\{p\}$. Then $\lambda_1' = R(p)$ is the only nonzero eigenvalue of $P'AP'$ and is a Poincaré-Rayleigh-Ritz upper bound for $\lambda_1$. The numbers $\lambda_1'' \leq \lambda_2'' \leq \cdots$ are the eigenvalues of the first intermediate problem

$$Au - (Au, p)p = \lambda u, \quad (u, p) = 0,$$

and satisfy the inequality

$$\lambda_j'' \leqq \lambda_{1+j}, \quad j = 1, 2, \cdots .$$

Applying our inequality (1), we have

$$0 \leqq \lambda_{1+j} - \lambda_j'' \leqq \lambda_1' - \lambda_1, \quad j = 1, 2, \cdots .$$

This inequality shows that the intermediate problem associated with a given vector $p$ yields *lower* bounds for the higher eigenvalues which are at least as accurate as the Poincaré-Rayleigh-Ritz *upper* bound for $\lambda_1$ obtained with the same $p$.

**4. Remark.** Let us point out here that Aronszajn's inequality (5) should not be confused with the following essentially different inequality due to Weyl [10, p. 445].

THEOREM. *Let $K$, $K^{(1)}$ and $K^{(2)}$ be compact self-adjoint operators on $H$ such that $K = K^{(1)} + K^{(2)}$. Then the nonpositive eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \cdots$, $\lambda_1^{(1)} \leqq \lambda_2^{(1)} \leqq \cdots$, and $\lambda_1^{(2)} \leqq \lambda_2^{(2)} \leqq \cdots$ of $K$, $K^{(1)}$, and $K^{(2)}$ satisfy the inequality*

$$\lambda_i^{(1)} + \lambda_j^{(2)} \leqq \lambda_{i+j-1}, \quad i, j = 1, 2, \cdots .$$

This inequality comes directly from a maximum-minimum principle, while Aronszajn's inequality (5) requires the above lemma, in addition to a maximum-minimum principle.

For recent expositions of the theory of intermediate problems see the books by G. Fichera [7] and S. H. Gould [8] and the paper of J. B. Diaz [9].

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INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS,
    UNIVERSITY OF MARYLAND