ON ČEBYŠEV SUBSPACES AND UNCONDITIONAL BASES IN BANACH SPACES

BY J. R. RETHERFORD[1]

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1. Introduction. Let E be a Banach space, Z a linear subspace of E and x an element of E. An element $z_0 \in Z$ is a *best approximation of x from Z* provided

$$\|x - z_0\| = \inf_{z \in Z} \|x - z\|.$$

Thus, to every linear subspace $Z \subset E$ and element $x \in E$ there corresponds a bounded closed convex (possibly empty) set

$$B_Z(x) = \{z_0 \in Z : \|x - z_0\| = \inf_{z \in Z} \|x - z\|\}.$$

Following Phelps [9] we say that $Z \subset E$ is a Čebyšev subspace if $B_Z(x)$ is one pointed for each $x \in E$.

If (x_i, f_i) is a Schauder basis for E, i.e. $(x_i) \subset E$, $(f_i) \subset E^*$, $f_i(x_j) = \delta_{ij}$ and $x = \sum_{i=1}^{\infty} f_i(x)x_i$ for each $x \in E$, let $L_n = [x_i \mid i \leq n]$, the linear span of $x_1, \cdots, x_n$ and let $L^n = [x_i \mid i > n]$, the closed linear span of $x_{n+1}, x_{n+2}, \cdots$. Also, let $s_n(x) = \sum_{i=1}^{n} f_i(x)x_i$ and $s^n(x) = \sum_{i=n+1}^{\infty} f_i(x)x_i = x - s_n(x)$. V. N. Nikol'skiĭ [7], [8] has shown that in a Banach space E with a Schauder basis, an equivalent norm can be given E such that, with respect to this new norm, both L_n and L^n are Čebyšev subspaces and, moreover,

$$B_{L_n}(x) = \{s_n(x)\} \qquad \text{and} \qquad B_{L^n}(x) = \{s^n(x)\}.$$

Now let (x_i, f_i) be an unconditional basis for E, i.e. a Schauder basis with the property that $x = \sum_{i=1}^{\infty} f_{p(i)}(x)x_{p(i)}$ for each permutation p of ω (the positive integers) and each $x \in E$. If $\sigma \in \Sigma$, the finite subsets of ω, let $L_\sigma = [x_i \mid i \in \sigma]$, $L^\sigma = \{x_i \mid i \in \omega \backslash \sigma\}$, $s_\sigma(x) = \sum_{i \in \sigma} f_i(x)x_i$ and $s^\sigma(x) = x - s_\sigma(x)$. Also, let $B_\sigma(x) = B_{L_\sigma}(x)$ and $B^\sigma(x) = B_{L^\sigma}(x)$.

Motivated by the fundamental work of Nikol'skiĭ mentioned above and by a theorem of Gelfand [3], Singer [10] showed that the norm, $|\ |$, defined by

$$|x| = \sup_{\sigma \in \Sigma} \left\| \sum_{i \in \sigma} f_i(x)x_i \right\| + \sum_{i=1}^{\infty} \|f_i(x)x_i\|/2^i$$

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is equivalent to the original norm of E and that with respect to $|\ |$, L^σ is a Čebyšev subspace and $B^\sigma = \{s^\sigma(x)\}$.

In [10] and later in [11] Singer raised the following problems: In a Banach space E with an unconditional basis, can E be given an equivalent norm making (i) L_σ a Čebyšev subspace with $B_\sigma(x) = \{s_\sigma(x)\}$ for each $x \in E$ and $\sigma \in \Sigma$? (ii) both L_σ and L^σ Čebyšev subspaces with $B_\sigma(x) = \{s_\sigma(x)\}$ and $B^\sigma(x) = \{s^\sigma(x)\}$ for each $x \in E$ and $\sigma \in \Sigma$?

The purpose of this note is to give an affirmative answer to (ii) (and hence to (i)).

Before proceeding to this result let us call attention to the interesting "Nikol'skiĭ" norm introduced by V. Istratescu [5].

2. The norm. M. M. Grinblyum [4] (see also M. M. Day [2]) has shown that if (x_i, f_i) is an unconditional basis for a Banach space E then there is a $K \geqq 1$ such that

$$\sup_{\sigma \in \Sigma} \left\| \sum_{i \in \sigma} f_i(x) x_i \right\| \leqq K \|x\| \quad \text{for each } x \in E. \tag{2.1}$$

2.2. Lemma. *Let (y_i) be a sequence in E and let (a_i) be a sequence of scalars. Then, for each $\sigma \in \Sigma$,*

$$\left\| \sum_{i \in \sigma} a_i y_i \right\| \leqq 4 \sup_{i \in \sigma} |a_i| \sup_{\beta \subset \sigma} \left\| \sum_{i \in \beta} y_i \right\| \cdot$$

Lemma 2.1 is proved in [6] in the generality of locally convex spaces and numerous applications of the inequality are given.

2.3. Corollary. *If (x_i, f_i) is an unconditional basis for E, (a_i) and (ϵ_i) sequences of scalars with $|\epsilon_i| \leqq 1$ then*

$$\left\| \sum_{i=1}^{n} \epsilon_i a_i x_i \right\| \leqq 4K \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

where K satisfies (2.1).

2.3 has been given previously by Bessaga and Pełczynskiĭ [1].

2.4. Lemma. *A series $\sum_{i=1}^{\infty} y_i$ in E is unconditionally convergent if and only if $\sum_{i=1}^{\infty} |f(y_i)|$ converges uniformly for $\|f\| \leqq 1$, $f \in E^*$.*

For a proof of 2.4 see [11]. Let us observe that the part of Lemma 3.4 we need follows immediately from Lemma 2.2.

2.5. Theorem. *Let (x_i, f_i) be an unconditional basis for a Banach space E (with norm $\|\ \|$) and for each $x \in E$ let*

$$|||x||| = \sup_{||f||\leq 1} \sum_{i=1}^{\infty} |f_i(x)f(x_i)| + \sum_{i=1}^{\infty} ||f_i(x)x_i||/2^i$$

then

(i) $||| \ |||$ *is norm on* E *equivalent to* $|| \ ||$, *in fact* $||x|| \leqq |||x||| \leqq 6K||x||$, *where* K *satisfies* (2.1),

(ii) *with respect to* $||| \ |||$, L_σ *and* L^σ *are Čebyšev subspaces with* $B_\sigma(x) = \{s_\sigma(x)\}$ *and* $B^\sigma(x) = \{s^\sigma(x)\}$ *for each* $x \in E$ *and* $\sigma \in \Sigma$.

PROOF. Let $f \in E^*$, $||f|| \leqq 1$. Choose ϵ_i, $|\epsilon_i| = 1$ such that

$$\sum_{i=1}^{\infty} |f_i(x)f(x_i)| = \left| f \sum_{i=1}^{\infty} \epsilon_i f_i(x)x_i \right|$$

then by 2.3 we have

$$\sum_{i=1}^{\infty} |f_i(x)f(x_i)| \leqq \left\| \sum_{i=1}^{\infty} \epsilon_i f_i(x)x_i \right\| \leqq 4K||x||.$$

Also, $||f_i(x)x_i|| \leqq 2K||x||$, whence

$$|||x||| \leqq 6K||x||.$$

Also,

$$||x|| = \sup_{||f||\leq 1} |f(x)| \leqq \sup_{||f||\leq 1} \sum_{i=1}^{\infty} |f_i(x)f(x_i)| \leqq |||x|||.$$

That $||| \ |||$ is indeed a norm is clear and (i) holds. To see that (ii) holds let $\sigma \in \Sigma$ and let $y = \sum_{i\in\sigma} b_i x_i \in L_\sigma$. Then for $x = \sum_{i=1}^{\infty} f_i(x)x_i \in E$

$$|||x - y||| = \sup_{||f||\leq 1} \left[\sum_{i\in\sigma} |(f_i(x) - b_i)f(x_i)| + \sum_{i\in\omega\setminus\sigma} |f_i(x)f(x_i)| \right]$$

$$(2.6) \qquad + \sum_{i\in\sigma} ||(f_i(x) - b_i)x_i||/2^i + \sum_{i\in\omega\setminus\sigma} ||f_i(x)x_i||/2^i$$

and it is clear that $\inf_{y\in L_\sigma} ||x-y||$ is attained when $y = s_\sigma(x) = \sum_{i\in\sigma} f_i(x)x_i$.

Also, if $i \in \sigma$ is such that $b_i \neq f_i(x)$ then the third term on the right of (2.6) is not zero and it follows that, for $y = \sum_{i\in\sigma} b_i x_i$, $|||x-y||| > |||x - s_\sigma(x)|||$; i.e.,

$$B_\sigma(x) = \{s_\sigma(x)\} \quad \text{for each } x \in E \text{ and } \sigma \in \Sigma.$$

A similar argument shows that $B^\sigma(x) = \{s^\sigma(x)\}$ for each $x \in E$ and $\sigma \in \Sigma$, and so (ii) holds.

With obvious modifications Theorem 2.5 extends easily to Fréchet spaces.

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Louisiana State University, Baton Rouge