ON THE SUMMABILITY OF THE DIFFERENTIATED FOURIER SERIES

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Dedicated to Professor A. Zygmund on the occasion of his 65th birthday

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A classical theorem of Fatou [2, p. 99] asserts that if $f \in L(0, 2\pi)$ and the symmetric derivative of f at x_0,

$$f_s'(x_0) = \lim_{h \to 0} [f(x_0 + h) - f(x_0 - h)]/2h$$

exists, then the differentiated Fourier series of f is Abel summable to $f_s'(x_0)$ at x_0, or equivalently, if $u(r, x) = a_0/2 + \sum (a_k \cos kx + b_k \sin kx) r^k$ is the associated harmonic function, then

$$\lim_{r \to 1-0} u_x(r, x_0) = f_s'(x_0).$$

Let us suppose that ϕ is a real nonnegative function on an interval to the right of the origin, that $\phi(0) = 0$, and that $\phi(t) = O(t)$ as $t \to 0$. We say that a set is ϕ-dense at a point p if

$$m(E^c \cap I)/\phi(m(I)) \to 0$$

as $m(I) \to 0$, I an interval containing p. If ϕ is the identity function, this reduces to ordinary metric density. In the case $\phi(t) = t^\alpha$, we will say that E is α-dense at p. Proceeding in a manner entirely analogous to the classical definition of approximate limit and derivative, we say that

$$\phi\text{-}\lim_{t \to t_0}{}_{ap}\, g(t) = a$$

if for every $\epsilon > 0$, $E_3 = \{t \mid |g(t) - a| < \epsilon\}$ is ϕ-dense at t_0, and we define the ϕ-approximate symmetric derivative,

$$\phi\text{-}f_{aps}'(x_0) = \phi\text{-}\lim_{h \to 0}{}_{ap}\, [f(x_0 + h) - f(x_0 - h)]/2h.$$

We restrict our attention here to the case of most immediate interest, α-density, and prove the following

THEOREM. *Suppose f is in $L(0, 2\pi)$, of period 2π, essentially bounded in a neighborhood of x_0, and, for some $\alpha \geqq 2$, $y = \alpha\text{-}f_{aps}'(x_0)$. Then the*

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differentiated Fourier series of f is Abel summable to y at x_0. The value 2 cannot be replaced by a smaller value nor can essentially bounded be replaced by integrable.

Ikegami [1] has shown that f_s' cannot be replaced by f_{ap}' in Fatou's theorem, even if f is bounded. He introduced

$$\alpha\text{-}f_{ap}'(x_0) = \alpha\text{-}\lim_{h\to 0ap} [f(x_0+h)-f(x_0)]/h$$

and attempted to show that, for bounded f, Fatou's theorem holds with this derivative if $\alpha > 4$. His argument, however, contains an error, and when it is corrected yields this result only for $\alpha > 5$.

Turning to the proof of our result, we may suppose that $x_0 = 0$, $f(0) = 0$, and also $\alpha\text{-}f_{aps}'(0) = 0$ as in the classical case [2, p. 100–101]. For the Poisson kernel,

$$P(r, t) = \frac{1}{2}\cdot\frac{1 - r^2}{1 - 2r\cos t + r^2},$$

we have the estimates

$$P(r, t) < C\eta/(\eta^2 + t^2), \qquad |P_t(r, t)| < C\eta t/(\eta^4 + t^4),$$

where $\eta = 1 - r$ and, throughout this paper, C will denote a positive constant not necessarily the same at each occurrence. The first estimate here is well known; the other may be obtained in a similar manner.

We may assume $\alpha = 2$, for if $\alpha\text{-}f_{aps}'(0)$ exists for some $\alpha > 2$, it also exists and has the same value for $\alpha = 2$.

There is a $\delta_0 > 0$ and an $M > 0$ such that $|f(x)| \leqq M$ a.e. in $(-\delta_0, \delta_0)$. Now

$$u_x(r, 0) = -\frac{1}{\pi}\int_0^\pi (f(t) - f(-t))P_t(r, t)dt$$

and, for any $\delta \in (0, \delta_0)$, we may partition the interval of integration into $(0, \delta)$, (δ, δ_0), and (δ_0, π), denoting the absolute values of the above integral over these intervals by ϑ_1, ϑ_2, and ϑ_3 respectively. We show that these values can be made arbitrarily small by choosing r sufficiently close to 1.

Clearly

$$\vartheta_2 \leqq 2M\int_\delta^\pi |P_t(r, t)|dt < C\eta\delta^{-2}$$

and

$$\mathcal{I}_3 \leqq \int_{\delta_0}^{\pi} |f(t) - f(-t)| \, |P_t(r, t)| \, dt$$
$$< C\eta \int_{\delta_0}^{\pi} \frac{|f(t) - f(-t)|}{t^3} \, dt < C\eta.$$

Given an $\epsilon > 0$, we set

$$E = \{t \mid |[f(t) - f(-t)]/2t| \geqq \epsilon\}.$$

Then

$$\mathcal{I}_1 \leqq \left| \int_{E \cap (0,\delta)} \cdots \right| + \left| \int_{E^c \cap (0,\delta)} \cdots \right| = \mathcal{I}_1' + \mathcal{I}_1''$$

and we have

$$\mathcal{I}_1'' \leqq \epsilon \int_0^{\delta} 2t |P_t(r, t)| \, dt < -2\epsilon \int_0^{\pi} tP_t(r, t) dt < C\epsilon$$

by an integration by parts.

The estimation of $\mathcal{I}_1'$ is somewhat more difficult.

We now choose δ such that, for $t \in (0, \delta)$,

$$m(E \cap (0, t)) < \epsilon t^2.$$

Let $t_1 = \delta$ and choose t_k, $k = 2, 3, \cdots$, in $(0, \delta)$, decreasing and converging to zero. We let $I_k = (t_{k+1}, t_k)$. Then

$$\mathcal{I}_1' < MC\eta \int_{E \cap (0,\delta)} t/(\eta^4 + t^4) dt$$
$$< C\eta \sum m(E \cap I_k) t_k/(\eta^4 + t_{k+1}^4) < C\eta\epsilon \sum t_k^3/(\eta^4 + t_{k+1}^4).$$

Now let $t_k = \delta/2^{k-1}$. It is easily verified that

$$2^7 \int_{I_k} t^2/(\eta^4 + t^4) dt > t_k^3/(\eta^4 + t_{k+1}^4)$$

for every k and, therefore,

$$\mathcal{I}_1' < C\eta\epsilon \int_0^{\infty} t^2/(\eta^4 + t^4) dt < C\epsilon.$$

Thus

$$|u_x(r, 0)| < C(\epsilon + \eta + \eta\delta^{-2}) < C\epsilon$$

if η is sufficiently small, the constant being independent of the choice of ϵ.

Suppose now that $\alpha \in [1, 2)$ and choose $\beta \in (\alpha, 2)$. Let $I_n = (1/2^n, 1/2^n + 1/2^{\beta n})$ and $E = \cup I_n$. Then if $1/2^n < t \leqq 1/2^{n-1}$, there exist positive constants C and C' such that

$$C/2^{\beta n} < m(E \cap (0, t)) < C'/2^{\beta n}$$

for every n. Thus $m(E \cap (0, t)) = o(t^\alpha)$ as $t \to 0$. If $f = \chi_E$, the characteristic function of E, then for sufficiently small $\epsilon > 0$,

$$\{t \mid |[f(t) - f(-t)]/2t| \geqq \epsilon\} = E$$

and so

$$\alpha = f'_{aps}(0) = 0.$$

For $0 < a < b < \pi/2$, it may be shown that

$$-\int_a^b P_t(r, t)dt > C\eta r \frac{(a + b)(b - a)}{\eta^4 + b^4}\,.$$

Thus, if $\eta = 2^{-k}$, we have

$$\begin{aligned} u_x(r, 0) &= -\frac{1}{\pi}\sum \int_{I_n} P_t(r, t)dt > -\frac{1}{\pi}\int_{I_{k+1}} P_t(r, t)dt \\ &> C2^{-(\beta+2)k}/(2^{-4k} + (2^{-(k+1)} + 2^{-\beta(k+1)})^4) \\ &> C2^{(2-\beta)k} \to \infty \end{aligned}$$

as $k \to \infty$, which shows that values of $\alpha < 2$ are inadmissible.

Finally suppose $\alpha \geqq 2$, $\beta > \alpha$, and define E as above. Now let $f = \sum 2^{(\beta-1)n}\chi_{I_n}$. Then $f \in L(0, 2\pi)$ and α-$f'_{aps}(0) = 0$. However,

$$\begin{aligned} u_x(r, 0) &> -\int_{I_{k+1}} 2^{(\beta-1)(k+1)} P_t(r, t)dt \\ &> C2^{(\beta-1)(k+1)} \cdot 2^{(2-\beta)k} = C2^k \to \infty \end{aligned}$$

as $k \to \infty$, which shows that the requirement of essential boundedness cannot be removed.

References

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