AN ABSTRACT FRAMEWORK FOR THE THEORY OF PROCESS OPTIMIZATION[1]

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Introduction. Ten years ago the development of a maximum principle as a necessary condition for optimality of some control problems began a new era for optimization theory. Since that time different maximum principles have been proposed and proved for a great variety of optimization problems. All these maximum principles and their proofs have a similar structure. The aim of the present paper is to give this unique structure independently of the particular characteristics of any one of these problems.

The present paper is a further addition to the trend started in Gamkrelidze [**1**] and [**2**], Halkin [**3**] and [**4**], Neustadt [**5**].

1. **Optimization problem.** We are given a set L, a mapping $f=(f_1, f_2, \cdots, f_k)$ from L into E^k and an integer m with $1 \leqq m \leqq k$. The problem is to find an $\hat{x} \in L$ which maximizes $f_1(\hat{x})$ subject to the constraints $f_i(\hat{x}) \geqq 0$ if $i=2, 3, \cdots, m$ and $f_i(\hat{x})=0$ if $i=m+1, \cdots, k$.

2. **Some assumptions.** The set L is a subset of a linear space X. There is a set $M \subset X$ which is an approximation of L around $\hat{x}$ and a mapping $h=(h_1, \cdots, h_k): X \rightarrow E^k$ which is an approximation of f around $\hat{x}$. We shall require that

(i) the set M is convex and $\hat{x} \in M$.

(ii) the functionals h_i are convex for $i=1, \cdots, m$ and linear-plus-a-constant for $i=m+1, \cdots, k$.

(iii) for any set $S=\text{co}\{\hat{x}, x_1, \cdots, x_l\} \subset M$ there is a mapping $\zeta: M \rightarrow L$ such that the mappings $f \circ \zeta$ and h are continuous over S (with respect to the usual finite dimensional topology on S) and "tangent at $\hat{x}$ over S" which means that for any $\epsilon > 0$ there is an $\eta \in (0, 1]$ with the property that $|f(\zeta(x)) - h(x)| \leqq \epsilon\delta$ if $\delta \in (0, \eta]$ and $x \in \text{co}\{\hat{x}, \hat{x}+\delta(x_1-\hat{x}), \cdots, \hat{x}+\delta(x_l-\hat{x})\}$.

3. **Maximum principle.** The purpose of the present paper is to prove that there exists real numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that

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$$(\alpha)\quad \sum_{i=1}^{k} |\lambda_i| > 0,$$

$$(\beta)\qquad \lambda_1 \geqq 0 \quad \text{for } i = 1, 2, \cdots, m,$$

$$(\gamma)\quad \sum_{i=1}^{k} \lambda_i h_i(\hat{x}) \geqq \sum_{i=1}^{k} \lambda_i h_i(x) \quad \text{for all } x \in M.$$

4. **Proof of the maximum principle.** There is no loss of generality by assuming that $\hat{x}=0$ and that $f(0)=0$. Let $K=\{(\alpha_1, \cdots, \alpha_k): \alpha_i>0,\ i=1, \cdots, m;\ \alpha_i=0,\ i=m+1, \cdots, k\}$. We have $K\cap f(L)=\emptyset$. We want to prove that K and $h(M)$ are separated. We shall assume that K and $h(M)$ are not separated and show that this leads to $K\cap f(L)\neq\emptyset$. If the sets $h(M)$ and K are not separated then, Step I, there exists a set $S=\text{co}\{0, x_1, \cdots, x_l\}\subset M$ such that

(i) $h(S)$ and K are not separated,
(ii) $l=k-m+1$,
(iii) $h_j(x_i)>0$ for $j=1, \cdots, m$ and $i=1, \cdots, l$.

Let $S^0 = S\sim\{0\}$. Then, Step II, there exists a $\sigma > 0$ such that $h(S^0)\subset\{\rho(\alpha_1, \cdots, \alpha_k): \rho\in(0, 1], \sigma\leqq\alpha_i\leqq 1/\sigma, i=1, \cdots, m; -1/\sigma \leqq\alpha_i\leqq 1/\sigma, i=m+1, \cdots, k\}$. For every $\delta\in(0, 1]$ let $S_\delta^0=\{\delta x: x\in S\sim\{0\}\}$. Then, Step III, there exists a $\beta\in(0, 1]$ such that $f_i(\zeta(x))>0$ if $i=1, \cdots, m$ and $x\in S_\beta^0$ where ζ is the mapping from S into L given by the definition of M. Then, Step IV, $f(\zeta(S))\cap K\neq\emptyset$ which implies $f(L)\cap K\neq\emptyset$. This concludes the proof of the Maximum Principle. Steps I, II and III correspond to elementary properties of convex sets and convex functions in a finite dimensional Euclidean space. Step IV is a consequence of Brouwer fixed point theorem.

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