MEASURES OF AXIAL SYMMETRY FOR OVALS[1]

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B. Grünbaum [2] has made a thorough report of the known results on measures of central symmetry for convex sets. We seek here to measure the degree of axial symmetry (axiality) of an oval K (a compact convex set in E^2 with interior points).

DEFINITION. A measure of axiality is a real-valued function f defined on the class of ovals such that

(i) $0 \leqq f(K) \leqq 1$;

(ii) $f(K)=1$ if and only if K has an axis of symmetry (is axial);

(iii) f is similarity-invariant.

Let ϕ be a direction in the plane, $k(\phi)$ a line normal to the direction ϕ, $b_\phi(K)$ the breadth of K in the direction ϕ, $\mathrm{Cv}(S)$ the convex hull of the set S, $\lambda_\phi(K)$ the "load curve" of K in the direction ϕ, (i.e., the set of midpoints of all chords of K in the direction ϕ), $[K]$ the area of K, $|K|$ the perimeter of K, and $K_{k(\phi)}$ the Steiner symmetrand of K with respect to the line $k(\phi)$.

The following measures of axiality are studied, and lower bounds are determined for them on the classes of arbitrary ovals (K), centrally symmetric ovals (K_c), and ovals of constant breadth (K_1):

$$f_1(K) = \max_\phi \{1 - b_\phi[\mathrm{Cv}(\lambda_\phi(K)]/b_\phi(K)\},$$

$$f_2(K) = \max_\phi \max_k (1/b) \int_0^b r(\phi, k, y)\, dy,$$

where $b=b_{\phi+\pi/2}(K)$, $k=k(\phi)$, and $r(\phi, k, y)$ is the ratio (taken $\leqq 1$) of the lengths of the two parts into which a chord $\gamma=\gamma(y)$ of K in the direction ϕ is divided by k ($r=0$ if $\gamma \cap k=\varnothing$),

$$f_3(K) = \max_{K'} \{[K']/[K]: K' \text{ is axial, and } K' \subseteq K\},$$

$$f_4(K) = \max_{K''} \{[K]/[K'']: K'' \text{ is axial, and } K \subseteq K''\},$$

$$f_5(K) = \max_{K'} \{|K'|/|K| : K' \text{ is axial, and } K' \subseteq K\},$$

$$f_6(K) = \max_{K''} \{|K|/|K''| : K'' \text{ is axial, and } K \subseteq K''\},$$

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$$f_7(K) = \max_{\phi} \max_{k} \{[K_{k(\phi)} \cap K]/[K]\},$$

$$f_8(K) = \max_{\phi} \max_{k} \{[K]/[\mathrm{Cv}(K_{k(\phi)} \cup K)]\},$$

$$f_9(K) = \max_{\phi} \max_{k} \{|K_{k(\phi)} \cap K| / |K|\},$$

$$f_{10}(K) = \max_{\phi} \max_{k} \{|K| / |\mathrm{Cv}(K_{k(\phi)} \cup K)|\},$$

$$f_{11}(K) = \max_{\phi} \max_{k} \{|K_{k(\phi)}| / |K|\}.$$

Lower bounds for these measures have been established as follows:

		K	K_c	K_1
f_1	$\geqq$	$1/2$[2]	$\sqrt{2}/2$[3]	$(2\sqrt{3}-3)^{1/2}$
f_2	$\geqq$	$1/4$	$2\log 2 - 1$	0.5474
f_3	$\geqq$	$5/8$[4]	$2(\sqrt{2}-1)$[5,2]	$8(2-\sqrt{3})/3$
f_4	$\geqq$	$1/2$	$\sqrt{2}/2$	$3(\pi-\sqrt{3})/4(3-\sqrt{3})$
f_5	$\geqq$	0.649	0.8045	$2\sqrt{2}/\pi$
f_6	$\geqq$	0.768	0.8045	$3\pi/8(3-\sqrt{3})$.

Lower bounds for the remaining measures are obtained from the facts that $f_i(K) \geqq f_{i-4}(K)$, $i=7, 8, 9, 10$, and $f_{11}(K) \geqq f_9(K)$ for every oval K. The only other special result not included in the above table is $f_{11}(K_1) \geqq (2-2\sqrt{3}/\pi)^{1/2}$.

Proofs of these results will be published elsewhere.

REFERENCES

1. G. D. Chakerian and S. K. Stein, *On the symmetry of convex bodies*, Bull. Amer. Math. Soc. **70** (1964), 594–595.

2. B. Grünbaum, *Measures of symmetry for convex sets*, Proc. Sympos. Pure Math., Convexity, Vol. 7, Amer. Math. Soc., Providence, R. I., 1963.

3. F. Krakowski, *Bemerkung zu einer Arbeit von W. Nohl*, Elem. Math. **18** (1963), 60–61.

4. W. Nohl, *Die innere axiale Symmetrie zentrischer Eibereiche der euklidischen Ebene*, Elem. Math. **17** (1962), 59–63.

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[2] Best possible lower bound.

[3] Conjecture; this is the g.l.b. on the class of parallelograms.

[4] Priority for this result must be given to F. Krakowski [**3**].

[5] Result of W. Nohl [**4**].