EXTREMAL LENGTH AND REMOVABLE BOUNDARIES OF RIEMANN SURFACES

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1. **Introduction.** Given a Riemann surface R let KD denote the space of harmonic functions u on R with finite Dirichlet norm $\|du\|$ and such that $*du$ is semiexact, i.e., $\int_c * du = 0$ for all dividing cycles c. Then O_{KD} denotes the class of Riemann surfaces R for which every function in KD is constant. Clearly $O_{HD} \subset O_{KD} \subset O_{AD}$ and for planar surfaces $O_{KD} = O_{AD}$. Under various names, this class O_{KD} has been studied by many authors (see, for example, Royden [4], Sario [5]).

The concept of the extremal length $\lambda(\mathfrak{F})$ of a family $\mathfrak{F}$ of curves on a Riemann surface R can be extended to the case that $\mathfrak{F}$ is a family of curves on the Kerékjártó-Stoilöw compactification $\hat{R}$ of R merely by eliminating the ideal points from each curve. Let α_0, α_1 be compact subsets of R. Define $\hat{\mathfrak{F}}$ to be the family of all arcs on $\hat{R}$ with initial point in α_0 and endpoint in α_1. Define $\mathfrak{F}$ to be the subfamily of $\hat{\mathfrak{F}}$ consisting of all arcs in R. We consider two notions for the extremal distance between α_0 and α_1, viz., define

$$\lambda(\alpha_0, \alpha_1) = \lambda(\mathfrak{F}), \qquad \hat{\lambda}(\alpha_0, \alpha_1) = \lambda(\hat{\mathfrak{F}}).$$

The aim of this note is to announce the following

THEOREM. *A necessary and sufficient condition that* $\lambda(\alpha_0, \alpha_1) = \hat{\lambda}(\alpha_0, \alpha_1)$ *for all compact subsets* α_0, α_1 *of* R *is that* $R \in O_{KD}$.

Our Theorem is reminiscent of the already classical result of Ahlfors-Beurling [1]:

A plane point set E is an AD-null set if and only if the removal of E does not change extremal distances.

The relationship between these results will be discussed in §3 below.

2. **Sketch of the proof.** The complete proof will appear in a forthcoming book [3]. The main steps in proving the necessity of the extremal distance condition are the following. (i) To construct functions u, $\hat{u}$ on R such that $\lambda(\alpha_0, \alpha_1) = \|du\|^{-2}$ and $\hat{\lambda}(\alpha_0, \alpha_1) = \|d\hat{u}\|^{-2}$, (ii) to show that $R \in O_{KD}$ implies $u = \hat{u}$. (Actually, these steps are applied to each component of $R - \alpha_0 - \alpha_1$, rather than R itself.)

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Step (i) was accomplished in [2]. There it is shown that if α_0, α_1, γ_0, γ_1 is an (admissible) partition of the ideal boundary of a Riemann surface S then a harmonic function $u(\alpha_0, \alpha_1, \gamma_0, \gamma_1)$ on S can be constructed which is determined by the following conditions—they are to be interpreted in the sense of a limit via an exhaustion of S: (1) $u(\alpha_0, \alpha_1, \gamma_0, \gamma_1) \equiv 0$ on α_0, (2) $u(\alpha_0, \alpha_1, \gamma_0, \gamma_1) \equiv 1$ on α_1, (3) $u(\alpha_0, \alpha_1, \gamma_0, \gamma_1)$ has vanishing normal derivative along γ_0 (L_0-behavior near γ_0), (4) along each component of γ_1, $u(\alpha_0, \alpha_1, \gamma_0, \gamma_1)$ is constant and has vanishing flux (L_1-behavior near γ_1). Furthermore, it is shown that $\|du(\alpha_0, \alpha_1, \gamma_0, \gamma_1)\|^{-2} = \lambda(\mathfrak{F}(\alpha_0, \alpha_1, \gamma_0, \gamma_1))$ where $\mathfrak{F}(\alpha_0, \alpha_1, \gamma_0, \gamma_1)$ is the family of arcs on $S \cup \alpha_0 \cup \alpha_1 \cup \gamma_1$ with initial point in α_0 and endpoint in α_1. Step (i) now follows since we have $u = u(\alpha_0, \alpha_1, \beta, \phi)$ and $\hat{u} = u(\alpha_0, \alpha_1, \phi, \beta)$ where β is the ideal boundary of R.

Step (ii) is accomplished by showing that on any $S \in O_{KD}$, L_0- and L_1-behavior cannot be distinguished.

To prove that the extremal distance condition of the Theorem is sufficient for $R \in O_{KD}$ we consider a consequence of the assumption $u = \hat{u}$ when α_0, α_1 vary over two systems of concentric disks centered at points ζ_0, ζ_1 of R. For $i = 1, 2$ let p_i denote a harmonic function on R with simple logarithmic poles at ζ_0 and ζ_1 of opposite sign, and with L_i-behavior near the ideal boundary β of R. As a limiting case of $u = \hat{u}$ we derive $p_0 = p_1 + \text{constant}$. In general, the differential $\psi = dp_1 - dp_0$ has the reproducing property

$$\iint_R dh \wedge * \psi = 2\pi \int_{\zeta_0}^{\zeta_1} dh$$

for all $h \in KD$. Since $\psi = 0$ in our case, $R \in O_{KD}$.

3. **Remarks.** Let R be a region in the extended plane $\boldsymbol{P}$ and let $E = \boldsymbol{P} - R$. Then $\lambda(\alpha_0, \alpha_1)$ is the usual extremal distance between α_0, α_1 on $\boldsymbol{P} - E$. Let $\lambda_{\boldsymbol{P}}(\alpha_0, \alpha_1)$ denote the extremal distance between α_0, α_1 on $\boldsymbol{P}$. Then we have

$$\lambda(\alpha_0, \alpha_1) \geqq \lambda_{\boldsymbol{P}}(\alpha_0, \alpha_1) \geqq \hat{\lambda}(\alpha_0, \alpha_1).$$

Thus our Theorem immediately implies the "only if" part of the Ahlfors-Beurling Theorem.

To derive the converse, assume that the removal of E does not change extremal distances. As in §2 we see that this implies that E is a removable singularity for the function p_0 defined on R. In general, the partial derivative of p_0 with respect to Re ζ_1 yields the real part of the horizontal slit mapping function for R with pole at ζ_1. It follows that E is a removable singularity for any parallel slit mapping. Thus the span of R vanishes; hence $R \in O_{AD}$.

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