HIGHER PRODUCTS

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W. S. Massey has defined a class of higher order cohomology operations of several variables, the higher products [2]. In this paper, we shall present a relativized definition of the higher products. We shall go on to list some of the algebraic and functorial properties of these operations. Finally, we shall describe a related cohomology operation of one variable. In certain cases, the latter operation can be computed in terms of primary Steenrod operations.

1. **Notation and definitions.** Throughout this paper, let $\overline{X}$ be a topological space and let (X_i, A_i) be pairs of subspaces of $\overline{X}$, for $i=1, \cdots, k$, such that $\bigcup_{r=1}^{k} A_r \subset \bigcap_{r=1}^{k} X_r$. Furthermore, for $1 \leqq i, j \leqq k$, assume that the triads $(\overline{X}, A_i, A_j)$ are excisive in the singular cohomology theory. This condition is satisfied if each X_i and A_i are open in $\overline{X}$ or if $\overline{X}$ is a CW complex and each X_i and A_i are subcomplexes. Let $u_1, \cdots, u_k$ be cohomology classes in the singular cohomology groups $H^{p_1}(X_1, A_1), \cdots, H^{p_k}(X_k, A_k)$ respectively, where the coefficients are in a fixed commutative ring R with identity. Finally, let $p(i, j) = \sum_{r=i}^{j} p_r - 1$ and $(X, A) = (\bigcap_{r=1}^{k} X_r, \bigcup_{r=1}^{k} A_r)$.

Under certain conditions, we may define the k-fold product $\langle u_1, \cdots, u_k \rangle$. Our definition shall be similar to the provisional definition of Massey [2].

DEFINITION 1. A defining system for $\langle u_1, \cdots, u_k \rangle$, A, is a set of singular cochains $(a_{i,j})$, for $1 \leqq i \leqq j \leqq k$ and $(i, j) \neq (1, k)$, satisfying the conditions:

(1.1) $a_{i,j} \in C^{p(i,j)+1}(\bigcap_{r=i}^{j} X_r, \bigcup_{r=i}^{r} A_r)$,

(1.2) $a_{i,i}$ is a cocycle representative of u_i, $i=1, \cdots, k$ and

(1.3) $\delta a_{i,j} = \sum_{r=i}^{j-1} (-1)^{(j+1-r)p(i,r)} a_{i,r}\, a_{r+1,j}$.

The related cocycle of A is the singular cocycle of $C^*(X, A)$

(1.4) $\sum_{r=1}^{k-1} (-1)^{(k+1-r)p(1,r)} a_{1,r}\, a_{r+1,k}$.

DEFINITION 2. The k-fold product $\langle u_1, \cdots, u_k \rangle$ is said to be defined if there is a defining system for it. If it is defined, then $\langle u_1, \cdots, u_k \rangle$ consists of all classes $w \in H^{p(1,k)+2}(X, A)$ for which there exists a defining system A whose related cocycle represents w.

If $k=2$, then the higher product $\langle u_1, u_2 \rangle$ is the ordinary cup product

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u_1u_2. If $k=3$, then $\langle u_1, u_2, u_3\rangle$ is defined if and only if the cup products $u_1u_2=0$ and $u_2u_3=0$. In this case the related cocycles are of the form $a_{12}a_{33}-(-1)^{p_1}a_{11}a_{23}$. This is a secondary operation, the Massey triple product as defined in [4].

The k-fold product is a $(k-1)$-order cohomology operation of k variables. In order for $\langle u_1, \cdots, u_k\rangle$ to be defined, it is necessary that the $(k-2)$-order operations $\langle u_1, \cdots, u_{k-1}\rangle$ and $\langle u_2, \cdots, u_k\rangle$ be defined and contain the zero element. In general this condition is not sufficient. There must exist defining systems A' and A'' for $\langle u_1, \cdots, u_{k-1}\rangle$ and $\langle u_2, \cdots, u_k\rangle$ respectively, for which not only do the related cocycles of each cobound but also $a'_{i,j}=a''_{i,j}$ for $1<i \leqq j<k$. In this case, we say that $\langle u_1, \cdots, u_{k-1}\rangle$ and $\langle u_2, \cdots, u_k\rangle$ vanish simultaneously.

2. **Properties.** We take the position that the higher products are analogous to the cohomology cup product. The properties listed below are generalizations of well-known relations satisfied by the cup product.

2.1. *Naturality.* For $i=1, \cdots, k$, let (Y_i, B_i) be pairs of subspaces of the topological space $\overline{Y}$ satisfying the conditions of §1. Let $g\colon \overline{Y}\to\overline{X}$ be a continuous map such that the image of (Y_i, B_i) under g is contained in (X_i, A_i) and denote by $g_i\colon (Y_i, B_i)\to(X_i, A_i)$ the induced map. Also, with $(Y, B)=(\bigcap_{r=1}^k Y_r, \bigcup_{r=1}^k B_r)$, let $\bar{g}\colon (Y, B)\to(X, A)$ be the induced map. If $\langle u_1, \cdots, u_k\rangle$ is defined, then so is $\langle g_1^*u_1, \cdots, g_k^*u_k\rangle$ and $\bar{g}^*\langle u_1, \cdots, u_k\rangle\subset\langle g_1^*u_1, \cdots, g_k^*u_k\rangle$.

2.2. *Scalar multiplication.* Assume that the product $\langle u_1, \cdots, u_k\rangle$ is defined. Then $\langle u_1, \cdots, xu_t, \cdots, u_k\rangle$ is defined for any $x\in R$, $t=1, \cdots, k$ and $x\langle u_1, \cdots, u_k\rangle\subset\langle u_1, \cdots, xu_t, \cdots, u_k\rangle$.

2.3. *Coboundary formula.* For some $t=1, \cdots, k$, assume that $(X_t, A_t)=(B, C)$ and $(X_i, A_i)=(Y, C)$ for $i\neq t$, where (Y, B, C) is a triple of topological spaces. If $\langle u_1, \cdots, u_t, \cdots, u_k\rangle$ is defined as a subset of $H^{p(1,k)+2}(B, C)$, then $\langle u_1, \cdots, \delta u_t, \cdots, u_k\rangle$ is defined as a subset of $H^{p(1,k)+3}(Y, B)$ and

$$\delta\langle u_1, \cdots, u_t, \cdots, u_k\rangle \subset (-1)^m\langle u_1, \cdots, \delta u_t, \cdots, u_k\rangle$$

with $m=\sum_{r=1}^{t-1} p_r+k$.

2.4. *Loop suspension.* Let $\pi\colon PX\to X$ be the path loop fibration over X. Then $E_A=\pi^{-1}(A)$ is the space of paths in X starting from the base point and ending in A. The relative loop suspension homomorphism $\sigma\colon H^n(X, A)\to H^{n-1}(E_A)$ is defined as the composite map

$$H^n(X, A) \xrightarrow{\pi^*} H^n(PX, E_A) \xleftarrow{\delta} H^{n-1}(E_A).$$

Assume that $\langle u_1, \cdots, u_k\rangle$ is defined as a subset of $H^{p(1,k)+2}(X, A)$. Then $\sigma\langle u_1, \cdots, u_k\rangle$ is the subset of $H^{p(1,k)+1}(E_A)$ consisting solely of the zero element.

2.5. *Associativity*. Let $\langle u_1, \cdots, u_k\rangle$ be defined as a subset of $H^{p(1,k)+2}(X, A)$ and let $v\in H^q(X', A')$, where (X', A') is also a pair of subspaces of $\overline{X}$. Then the k-fold product $\langle u_1, \cdots, u_tv, \cdots, u_k\rangle$ is defined for each $t=1, \cdots, k$ as a subset of $H^{p(1,k)+q+2}(X\cap X', A\cup A')$ and satisfies the relations

$$\langle u_1, \cdots, u_k\rangle v \subset \langle u_1, \cdots, u_kv\rangle,$$
$$v\langle u_1, \cdots, u_k\rangle \subset (-1)^{kq}\langle vu_1, \cdots, u_k\rangle$$

and

$$\langle u_1, \cdots, u_tv, u_{t+1}, \cdots, u_k\rangle \cap \langle u_1, \cdots, u_t, vu_{t+1}, \cdots, u_k\rangle \neq \varnothing.$$

These relations may be interpreted as equalities modulo the sum of the indeterminacies.

2.6. *Symmetry*. Assume that the higher product $\langle u_1, \cdots, u_k\rangle$ is defined. Then the symmetric product $\langle u_k, \cdots, u_1\rangle$ is also defined and $\langle u_1, \cdots, u_k\rangle = (-1)^n\langle u_k, \cdots, u_1\rangle$ with $n = \sum_{1\leq r<s\leq k} p_rp_s + (k-1)(k-2)/2$.

2.7. *Permutability*. Assume that all the k-fold products $\langle u_t, \cdots, u_k, u_1, \cdots, u_{t-1}\rangle$ are defined simultaneously as subsets of $H^{p(1,k)+2}(X, A)$. Then there are classes $w_t\in\langle u_t, \cdots, u_{t-1}\rangle$, for $t=1, \cdots, k$, such that $\sum_{t=1}^{k} (-1)^{t(k+1)+\pi(t)}w_t=0$, where $\pi(1)=0$ and $\pi(t)=(p_1+ \cdots +p_{t-1})(p_t+ \cdots +p_k)$ for $t>1$.

The proofs of these formulas and relations are computational in nature. For the proof of 2.5, we use the $\cup_1$-product of Steenrod [3] and a formula of G. Hirsch [1]. The formulas 2.6 and 2.7 require the use of a set of "commuting" chain homotopies which we may construct by means of the acyclic model theorem.

3. **The operation** $\langle u\rangle^k$. If we assume that $u_1=u_2= \cdots =u_k =u\in H^m(X, A)$, then we can define a related higher order cohomology operation $\langle u\rangle^k$ with less indeterminacy.

DEFINITION 1'. A defining system for $\langle u\rangle^k$, A^*, is a set of singular cochains (a_n), for $n=1, \cdots, k-1$, satisfying the conditions:

(3.1) $a_n\in C^{n(m-1)+1}(X, A)$,

(3.2) a_1 is a cocycle representative of u, and

(3.3) $\delta a_n = \sum_{r=1}^{n-1} (-1)^{rn(m-1)}a_r\, a_{n-r}$.

The related cocycle of A^* is the singular cocycle of $C^*(X, A)$

(3.4) $\sum_{r=1}^{k-1} (-1)^{rk(m-1)}a_ra_{k-r}$.

DEFINITION 2′. The operation $\langle u\rangle^k$ is said to be defined if there is a defining system for it. If it is defined, then $\langle u\rangle^k$ consists of all classes $w\in H^{k(m-1)+2}(X, A)$ for which there exists a defining system A^* whose related cocycle represents w.

If $\langle u\rangle^k$ is defined, then so is the k-fold product $\langle u, \cdots, u\rangle$ and $\langle u\rangle^k\subset\langle u, \cdots, u\rangle$. Also $\langle u\rangle^k$ is defined if and only if $\langle u\rangle^{k-1}$ is defined and contains the zero class.

Let p be an odd prime and let β be the Bockstein operator associated with the exact sequence of coefficient groups $0\rightarrow Z_p\rightarrow Z_{p^2}\rightarrow Z_p\rightarrow 0$. Furthermore, let P^m be the Steenrod pth power operation,

$$P^m: H^q(X; Z_p) \rightarrow H^{q+2m(p-1)}(X; Z_p).$$

THEOREM A. *If* $u\in H^{2m+1}(X; Z_p)$, *then* $\langle u\rangle^p$ *is defined as a single class in* $H^{2mp+2}(X; Z_p)$ *and* $\langle u\rangle^p = -\beta P^m u$.

If u is a one-dimensional class (mod p) for any prime p, then we may completely characterize the operation $\langle u\rangle^k$ by the following theorem.

THEOREM B. *Let* $\iota\in H^1(Z_{p^n}; Z_p)$ *be the* mod p *reduction of the fundamental class* ι_n *of* $H^1(Z_{p^n}; Z_{p^n})$. *Then* $\langle\iota\rangle^{p^n}$ *is defined as the single class* $-\beta_n\iota_n\in H^2(Z_{p^n}; Z_p)$, *where* β_n *is the Bockstein coboundary operator associated with the exact sequence of coefficient groups*

$$0 \rightarrow Z_p \rightarrow Z_{p^{n+1}} \rightarrow Z_{p^n} \rightarrow 0.$$

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