RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable.

GENERALIZED INTERPOLATION BY ANALYTIC FUNCTIONS[1]

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1. Generalized interpolation by entire functions.

DEFINITION 1. Let S be a set of complex numbers. The distance from a point z to S is denoted by $d(z, S)$ i.e.,

$$d(z, S) = \underset{s \in S}{\text{g.l.b.}} \ |z - s|. \tag{1}$$

THEOREM 1. *Let S be a set of complex numbers such that*

$$d(z, S) \leqq |z|^{1-\epsilon} \tag{2}$$

for some $\epsilon > 0$ and all sufficiently large $|z|$. Let $\{z_h\} = \{z_1, z_2, \cdots\}$ be a sequence of complex numbers without finite limit points, then there exist entire functions $F(z)$ with $F^{(m)}(z_h) \in S$ for $m = 0, 1, 2, \cdots$; $h = 1, 2, 3, \cdots$. The set of such functions has the power of the continuum, even when a finite number of the values $F^{(m)}(z_h)$ are prescribed arbitrarily in S.

COROLLARY 1. *There exists a nondenumerable set of transcendental entire functions, which together with all their derivatives, assume Gaussian integers or Gaussian primes etc. at all Gaussian integers.*

If $\{z_h\}$ consists of real points and S is a set of real numbers such that $d(x, S) \leqq |x|^{1-\epsilon}$ $(\epsilon > 0)$ for all sufficiently large $|x|$, then generalized interpolation by real entire function is possible which gives the following

COROLLARY 2. *There exists a nondenumerable set of transcendental*

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entire functions, which together with all their derivatives, assume rational primes at all rational integers.

2. **Generalized interpolation by analytic functions.** The Weierstrass factorization theorem and the Mittag-Leffler theorem can be generalized to noncompact Riemann surfaces. Consequently, we have the following generalization of Borel's interpolation.

THEOREM 2. *Given a noncompact Riemann surface R and a differential operator on R. Let $\{z_h\}$ be a sequence of points in R without limit points in R. Given a sequence of complex numbers $\{\alpha_{hl}\}$ where $l=0, 1, 2, \cdots, m_h-1$; $h=1, 2, 3, \cdots$, then there exists a function $\phi(z)$ analytic in R so that*

$$\phi^{(l)}(z_h) = \alpha_{hl} \tag{3}$$

for $l=0, 1, 2, \cdots, m_h-1$; $h=1, 2, 3, \cdots$.

This generalized Borel interpolation is used to prove the following

THEOREM 3. *Given a noncompact Riemann surface R, a differential operator on R, and a sequence of points $\{z_h\}$, $z_h \in R$, without limit points in R. Let S be a set of complex numbers such that*

$$d(z, S) \leqq |z|^{1-\epsilon} \tag{4}$$

for some $\epsilon > 0$ and all sufficiently large $|z|$, then there exist functions $F(z)$ analytic in R with $F^{(m)}(z_h) \in S$ for $m=0, 1, 2, \cdots$; $h=1, 2, 3, \cdots$. The set of such functions has the power of the continuum, even when a finite number of the values $F^{(m)}(z_h)$ are prescribed arbitrarily in S.

3. **Examples in which generalized interpolation is not possible.** We give several examples indicating that some of the limitations imposed on the set of values S and the sequence of points $\{z_h\}$ are necessary. If S is discrete, then the condition that the sequence $\{z_h\}$ has no cluster points in R is obviously necessary in order to have an interpolation by a nonconstant analytic function. It is also clear, by the example $S=\{0\}$, that some restriction on S is necessary.

The following lemmas, theorems and corollaries give less trivial examples which show that a generalized interpolation by nonconstant analytic functions is not always possible, even when S is a rather big portion of the set of all complex numbers.

LEMMA 1. *If an analytic function $f(z)$ has all its derivatives at two points in a finite set S, then $f(z)$ satisfies a differential equation of the form $f^{(m)}(z) = f^{(n)}(z)$ for some $m < n$ where n is bounded by a bound depending only on S, and the set of values $\{f^{(k)}(\alpha) \mid k=0, 1, 2, \cdots\}$ is therefore finite for any point α.*

LEMMA 2. *Given a finite set S and $z_0 \neq z_1$, then there is at most a finite set of analytic functions $f(z)$ so that*

$$f^{(m)}(z_0) \in S, \qquad f^{(m)}(z_1) \in S \qquad (m = 0, 1, 2, \cdots).$$

LEMMA 3. *Given a finite set S and a point z_0, then the set of points z_1, so that there exists a nonconstant analytic function $f(z)$ with $f^{(m)}(z_0) \in S$, $f^{(m)}(z_1) \in S$ $(m=0, 1, 2, \cdots)$ is denumerable.*

THEOREM 4. *Given a finite set S and two points z_1, z_2. There does not exist, in general, an analytic function which together with all its derivatives assumes the values in S at the points z_1 and z_2. In fact, it exists only for a denumerable set of values of $z_1 - z_2$.*

THEOREM 5. *Let S_k be the set of the first k positive integers, i.e., $S_k = \{1, 2, 3, \cdots, k\}$. If $k > 1$, then there is no analytic function which together with all its derivatives maps S_k into itself.*

THEOREM 6. *Let $\{z_1, z_2, z_3\}$ be such that the imaginary parts of $z_3 - z_1$ and $z_3 - z_2$ are incommensurable, then there is no nonconstant entire function which together with all its derivatives assumes real values at z_1, z_2 and z_3.*

THEOREM 7. *Let S be the set of complex numbers having positive real part, then there exists a sequence $\{z_h\}$ having arbitrarily small asymptotic density, such that there is no entire function which together with all its derivatives assumes values in S at all points of $\{z_h\}$.*

THEOREM 8. *Let S be a bounded set of complex numbers and $\{z_h\}$ be a sequence for which $d(z, \{z_h\}) < c \log |z|$ for some $c > 0$ and all $|z| > 2$ then there is no nonconstant analytic function which together with all its derivatives assumes values in S at all points of $\{z_h\}$.*

COROLLARY 3. *If the values of an analytic function and all its higher derivatives at every Gaussian integer form a bounded set of complex numbers, then the function is constant.*

4. Indication of proof.

PROOF OF THEOREM 1. We wish to construct a generalized interpolation series

$$F(z) = \sum_{n=0}^{\infty} a_n \prod_{\iota=1}^{H} (z - z_\iota)^{m_\iota}, \tag{5}$$

where the product in each term is finite, each exponent $m_1, m_2, \cdots, m_H$ is a nondecreasing function of n and $n = m_1 + m_2 + \cdots + m_H$. Thus as n is increased to $n+1$, there is exactly one m_h (possibly

$m_h=0$) which is increased to m_h+1 and we get

$$F^{(m_h)}(z_h) = a_n m_h! \prod_{i\neq h}^{H} (z_h - z_i)^{m_i} + X_{n-1}, \tag{6}$$

where the X_{n-1} involves a_ν with $\nu<n$. Thus we can determine the a_n successively by choosing $F^{(m_h)}(z_h)$ to be an element of the set S which is the nearest to X_{n-1}. Then we get

$$|a_n| \leqq \frac{d(X_{n-1}, S)}{m_h! \prod_{i\neq h} (z_h - z_i)^{m_i}} \cdot \tag{7}$$

We can now determine the m_i as functions of n, so that (5) converges uniformly in every circle $|z|\leqq r$.

PROOF OF THEOREM 2. Exactly analogous to the case of entire functions.

PROOF OF THEOREM 3. By the generalized Borel interpolation there exists a function $u=u(z)$, analytic in R such that (1) $u'(z_h)=1$ $(h=1, 2, \cdots)$, (2) $u(z_k)\neq u(z_l)$ for $k\neq l$, (3) $u(z_h)\to\infty$ as $n\to\infty$. We construct a generalized interpolation series of the form

$$F(z) = \sum_{n=0}^{\infty} a_n \prod_{i=1}^{H} (u(z) - u(z_i))^{m_i}$$

as in the case of Theorem 1.

PROOF OF LEMMA 1. Without loss of generality, let the two points be 0 and $z_0\neq 0$, assume that $f(z)=\sum_{n=0}^{\infty} s_n z^n/n!$ then $s_n\in S$. If we take m large enough, then $f^{(k)}(z_0)\in S$ $(k=0, 1, 2, \cdots)$ implies that s_{k+m+1} is determined uniquely by the finite sequence $\{s_k, s_{k+1}, \cdots, s_{k+m}\}$ and there are numbers p and n_0 such that $s_{n+p}=s_n$ for all $n\geqq n_0$.

PROOF OF LEMMA 2. All such functions satisfy linear differential equation of bounded order, and the function is uniquely determined by the finitely many choices of the initial values from the finite set S.

PROOF OF LEMMA 3. $f(z)$ satisfies one of the denumerable set of linear differential equations with initial values in S. Each function attains values in S only at a denumerable set of points.

PROOF OF THEOREM 4. Consequence of Lemma 3.

PROOF OF THEOREM 5. Since S_k is bounded, the function must be an entire function and its order must be $\rho\leqq 1$. Since S_k is a set of integers, we have $\rho\geqq k>1$ or the function is a polynomial. Since there is no such polynomial the contradiction is proved. (For $k=1$, there is the unique example $f(z)=e^{z-1}$.)

PROOF OF THEOREM 6. Let $f(z)=\sum_{n=0}^{\infty} a_n^{(k)}(z-z_k)^n$ $(k=1, 2, 3)$

then $a_n^{(k)}$ are real and $f(z)$ assumes real values on three lines through z_k $(k=1, 2, 3)$ parallel to the real axis. Using the Schwarz reflection principle repeatedly, we see that $f(z)$ assumes real values at all z.

PROOF OF THEOREM 7. The essential fact here is that S forms a convex cone and hence a semi-group under addition. Select $\{z_h\}$ so that $\mathrm{Im}(z_h)$ form a dense set and there are arbitrarily large $-\mathrm{Re}(z_h)$ for each $\mathrm{Im}(z_h)$. Then every $f(z)$ with $f^{(n)}(z_h)\in S$, $n=0, 1, 2, \cdots$, $h=1, 2, \cdots$ would satisfy $\mathrm{Re}\, f(z)>0$ for all z which is impossible.

PROOF OF THEOREM 8. Let $|z|>2$ and let z_h be a point for which $|z-z_h|<c\log|z|$. Then

$$\begin{aligned} |f(z)| &\leqq \sum_{n=0}^{\infty} |a_n^{(h)}|\, |z-z_h|^n/n! \\ &\leqq A\sum |z-z_h|^n/n! < A\,|z|^c. \end{aligned}$$

Thus, by Liouville's Theorem $f(z)$ is a polynomial, and unless $f(z)$ is a constant its values satisfy $|f(z)|\leqq A$ only on a bounded set and not on the whole sequence $\{z_h\}$.

PROOF OF COROLLARIES 1, 2, 3. For Gaussian integers S, $d(z, S)\leqq 2^{-1/2}$. For rational primes S, $d(x, S)<|x|^{(48+\epsilon)/77}$ $(\epsilon>0)$. For Gaussian primes S, we have $d(z, S)<|z|^{\phi}$ for some $\phi<1$.

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