A SIMPLE PROOF OF THE RABIN-KEISLER THEOREM

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For terminology and notation we refer to the two relevant papers of Rabin [3] and Keisler [1]. The following theorem is proved in [1] and is an improvement of the main result of [3].

THEOREM (RABIN-KEISLER). *Let α be an infinite nonmeasurable cardinal. Then every model of power α has a proper elementary extension of the same power if and only if $\alpha = \alpha^\omega$.*

The simple proof referred to in the title does not require the elaborate apparatus of limit ultrapowers (see [1]) or the generalized continuum hypothesis and that α be accessible (see [3]). On the other hand, the proof owes much to certain ideas in [3] and Keisler [2].

One direction of the theorem follows easily from elementary properties of ultrapowers. The following lemma will establish the other direction.

LEMMA. *Suppose α is an infinite nonmeasurable cardinal, $\mathfrak{M} = \langle A, R, S, \cdots \rangle$ is the complete model over a set A of power α, and $\mathfrak{M}' = \langle A', R', S', \cdots \rangle$ is a proper elementary extension of $\mathfrak{M}$. Then $|A'| \geqq \alpha^\omega$.*

PROOF. By a well-known result in set theory (using finite sequences of elements from A), there exists a family

$$P = \{P_\beta : \beta < \alpha^\omega\}$$

of countably infinite subsets P_β of A such that $|P| = \alpha^\omega$ and $P_\beta \cap P_\gamma$ is finite whenever $\beta \neq \gamma$. Well-order each P_β,

$$P_\beta = \{p_{\beta n} : n < \omega\}.$$

Let $x \in A' - A$, and let

$$D = \{Q : Q \subset A \text{ and } x \in Q'\}.$$

It is easily seen that D is a nonprincipal ultrafilter over A. By hypothesis D is countably incomplete. Hence, there exists a strictly decreasing sequence

$$A = Q_0 \supset Q_1 \supset \cdots \supset Q_n \supset \cdots$$

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of sets $Q_n \in D$ such that $\bigcap_n Q_n = 0$. Fix $\beta < \alpha^\omega$. Define a function F_β mapping A onto P_β as follows: for each $a \in A$,

$$F_\beta(a) = p_{\beta n} \text{ if and only if } a \in Q_n - Q_{n+1}.$$

Notice that the function F_β (considered as a binary relation) and the sets P_β, Q_n are among the relations listed in $\mathfrak{M}$. Since $\mathfrak{M} \prec \mathfrak{M}'$, it follows that F'_β is a function mapping A' onto P'_β. Furthermore, for each $a' \in A'$,

$$F'_\beta(a') = p_{\beta n} \text{ if and only if } a' \in Q'_n - Q'_{n+1}.$$

Since $x \in Q'_n$ for all n, we have

$$F'_\beta(x) \in P'_\beta - P_\beta.$$

Using the fact that $P_\beta \cap P_\gamma$ is finite whenever $\beta \neq \gamma$, we have $(P_\beta \cap P_\gamma)' = P'_\beta \cap P'_\gamma = P_\beta \cap P_\gamma$. Hence

$$F'_\beta(x) \neq F'_\gamma(x), \quad \text{whenever} \quad \beta \neq \gamma.$$

So $|A'| \geqq \alpha^\omega$ and the lemma is proved.

References

1. H. J. Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc. 107 (1963), 382–408.
2. ———, *Extending models of set theory* (Abstract), J. Symbolic Logic (to appear).
3. M. Rabin, *Arithmetical extensions with prescribed cardinality*, Indag. Math. 21 (1959), 439–446.

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