RESEARCH ANNOUNCEMENTS

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THE THEOREM OF THE THREE CLOSED GEODESICS

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1. The theorem we are referring to states that on every compact riemannian manifold M there exist three simple closed geodesics of which the energy is bounded in terms of a map of a sphere into M, cf. §6 for a precise formulation: there we also explain why this is the best possible general solution of the problem to give a lower bound for the number of simple closed geodesics of bounded energy on a compact riemannian manifold.

We call a geodesic simple if it does not cover another geodesic.

The theorem is a simultaneous generalization of the theorem of Lusternik-Schnirelmann [4] that on a surface of the type of the 2-sphere S^2 there are three simple geodesics, and of the theorem of Fet [2] that on every compact riemannian manifold M there is one simple closed geodesic.

We obtain this theorem, and several other new results on the existence of closed geodesics, from a refinement of the previously employed methods for studying the space of closed curves and singling out among these the geodesics. For a brief historical survey we refer to our note [3].

2. Our approach uses substantially the Morse theory on infinite dimensional manifolds as developed recently by Palais and Smale [7], [8], [9]. In our case, the infinite dimensional manifold is the space $\Lambda(M)$ of absolutely continuous maps $f=(f(t))$ of the parametrized circle $S^1=[0,1]/\{0,1\}$ into M which have, in any local chart, square integrable derivatives.

$\Lambda(M)$ is called *the space of parametrized closed curves on M*. It is a manifold modeled after a separable Hilbert space, cf. Palais [7]. The homotopy type of $\Lambda(M)$ depends only on the homotopy type of M.

The riemannian metric $\langle\ ,\ \rangle$ on M determines on $\Lambda(M)$ the energy

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function by $E(f)=\frac{1}{2}\int_0^1\langle f'(t),f'(t)\rangle dt$ and the riemannian metric $\langle\langle\ ,\ \rangle\rangle$ by

$$\langle\langle X, X\rangle\rangle = \int_0^1 (\langle X(t), X(t)\rangle + \langle DX(t)/dt, DX(t)/dt\rangle)dt$$

where $X=(X(t))$ is a tangent vector at a point $f=(f(t))$ of $\Lambda(M)$.

E is differentiable. $\Lambda(M)$ is complete with respect to the metric $\langle\langle\ ,\ \rangle\rangle$. With the help of this metric we define on $\Lambda(M)$ the vector field $-\text{grad}\, E$. The condition (C) of Palais and Smale is satisfied which allows to develop the Morse theory of the function E on $\Lambda(M)$, cf. [7] and [9].

A point $f\in\Lambda(M)$ is critical with respect to E, i.e., grad $E(f)=0$, if and only if f is a geodesic, parametrized proportional to arc length. Note that the point curves on M are critical points of $\Lambda(M)$. They form a nondegenerate critical submanifold $\Lambda^0(M)$ of index 0 which is isomorphic to M.

3. As additional feature we have on $\Lambda=\Lambda(M)$ the left action of the orthogonal group $O(2)$ which is induced from the usual action of $O(2)$ on the circle S^1. This action is continuous, it leaves the energy function E invariant and it is isometric with respect to $\langle\langle\ ,\ \rangle\rangle$. It follows that also the vector field $-\text{grad}\, E$ is invariant under the action of $O(2)$.

We introduce the quotient space $\Pi(M)=\Lambda(M)/O(2)$. Let π be the quotient map $\Lambda(M)\to\Pi(M)$. $\Pi=\Pi(M)$ is called the *space of unparametrized closed curves on* M. The homotopy type of $\Pi(M)$ depends only on the homotopy type of M.

Put $\pi\Lambda^0=\Pi^0$; this is isomorphic to Λ^0.

Let $\phi_t:\Lambda(M)\to\Lambda(M)(t\geqq 0)$ be the one-parameter transformation semigroup induced from the integration of the vector field $-\text{grad}\, E$. ϕ_t carries orbits of $O(2)$ into orbits. Hence, it induces a one-parameter transformation semigroup $\psi_t:\Pi(M)\to\Pi(M)$, satisfying $\psi_t\circ\pi=\pi\circ\phi_t$.

4. When talking about homology and cohomology, we always mean the singular theory with Z_2 coefficients.

Let u be a singular cycle of Π mod Π^0. Then $c(u)=\lim_{t\to\infty}\max_{f\in u}\psi_t(u)$ exists. Let z be a homology class of Π mod Π^0. Then also $c(z)=\inf_{u\in z}c(u)$ exists. $c(z)$ is called the *critical value* of z.

There exists a critical point on $\Lambda(M)$ of energy $c(z)$; $c(z)>0$ if $z\neq 0$. Hence, by taking the underlying simple closed geodesic, we find that for $z\neq 0$ there exists a simple closed geodesic of energy $\leqq c(z)$ on M.

There arises the question when two different homology classes z and z' in this way give rise to two different simple closed geodesics. To formulate a criterion we introduce the following concepts:

A homology class $z = z_k \in H_k(\Pi, \Pi^0)$ is called *subordinated* to the class $z' = z_{k+l} \in H_{k+l}(\Pi, \Pi^0)$ if both are nonzero and $l > 0$ and if there is a cohomology class $\zeta^l \in H^l(\Pi - \Pi^0)$ such that $z_k = z_{k+l} \cap \zeta^l$. Here we use the fact that there is a natural pairing $H_*(\Pi, \Pi^0) \otimes H^*(\Pi - \Pi^0) \to H_*(\Pi, \Pi^0)$ given by the cap product. The concept of subordination was introduced in this connection by Lyusternik [5] and used also by Al'ber [1].

Let $u = \{b: (K, K^0) \to (\Pi, \Pi^0)\}$ be a singular cycle of Π mod Π^0, K being a complex. We say that u has *local cross sections in* Λ if every point of $K - K^0$ is contained in an open set U of K for which there exists an equivariant map $a_{\mathfrak{U}}: \mathfrak{U} \times O(2) \to \Lambda$ which induces $b \mid \mathfrak{U}$.

Then we may state as the fundamental result of this paper the following

LEMMA. *Let z and z' be homology classes of* $\Pi(M)$ mod $\Pi^0(M)$ *such that z is subordinated to z' and z as well as z' can be represented by cycles possessing local cross sections in $\Lambda(M)$. Then there exist on M two simple closed geodesics of energy $\leqq c(z')$.*

Hence, we get as many simple closed geodesics on M as we can find pairwise subordinated homology classes in $\Pi(M)$ mod $\Pi^0(M)$ which can be represented by cycles possessing local cross sections in $\Lambda(M)$.

5. Let S be an irreducible compact symmetric space of rank 1, i.e., a sphere or a projective space. As was indicated already in [3], we have in $\Pi = \Pi(S)$ the subspace $C = C(S)$ of circles. Let $C^0 = C^0(S)$ be the subspace of point circles, isomorphic to S. Then we noted in [3] that the inclusion $i: C$ mod $C^0 \to \Pi$ mod Π^0 is injective in homology, and the same is true also for the inclusion $i: C - C^0 \to \Pi - \Pi^0$. Note that the cycles of $\Pi(S)$ which lie in $C(S)$ possess local cross sections in $\Lambda(S)$.

One can now show that if $h: S \to M$ is a homotopy equivalence, then there exist maps $H_*(\Pi(S), \Pi^0(S)) \to H_*(\Pi(M), \Pi^0(M))$ and $H_*(\Pi(S) - \Pi^0(S)) \to H_*(\Pi(M) - \Pi^0(M))$ which are bijective and which carry cycles lying in the space of circles $C(S)$ into cycles having local cross sections in $\Lambda(M)$.

So all that remains to be done for finding a lower bound for the number of simple closed geodesics on a space M of the homotopy type of S is to determine the maximal number of pairwise subordinated classes on the space of circles on S. This amounts to computing the cup length of the ring $H^*(C(S) - C^0(S)) = H^*(G(S))$ where $G(S)$ is the space of great circles on S.

The results are:

THEOREM 1. *On a compact riemannian manifold* $M=M^n$ *of the homotopy type of the sphere* S^n *there exist* $2n-s-1$ *simple closed geodesics, with* $0\leqq s=n-2^h<2^h$. *The energy of these geodesics is bounded in terms of a homotopy equivalence* $h\colon S^n\to M$.

Recall that the other compact irreducible symmetric spaces of rank 1 are the projective spaces $P^m(\lambda)$ over the reals ($\lambda=1$), the complex numbers ($\lambda=2$), the quaternions ($\lambda=4$) or the Cayley numbers ($\lambda=8$), of real dimension $n=m\lambda\geqq 2\lambda$ and $n=2\lambda=16$ for $\lambda=8$.

THEOREM 2. *On a compact riemannian manifold* $M=M^n$ *of the homotopy type of the projective space* $P^m(\lambda)$, $n=m\lambda$, *there exist at least* $2n-(2\lambda-1)s-1$ *simple closed geodesics, with* $0\leqq s=m-2^h<2^h$. *The energy of these geodesics is bounded in terms of a homotopy equivalence* $h\colon P^m(\lambda)\to M$.

6. THEOREM 3. *On a compact riemannian manifold* $M=M^n$ *there exist at least three simple closed geodesics. For* $\pi_1(M)=0$, *the energy of these geodesics is bounded in terms of a map of a k-sphere into* M, $2\leqq k\leqq n$.

REMARK. This is the best possible general result, as is shown by the following example due to Morse [6]. Given an arbitrarily large (and not too small) real number c, there exists a 2-dimensional ellipsoid E^2 with three different axes, all having their length close to 1, such that the only simple closed geodesics on E^2 of energy $\leqq c$ are the three principal ellipses. But these are just the three simple closed geodesics which are obtained, in the manner described below, from the Gauss map $h\colon S^2\to E^2$.

For an indication of the proof we restrict ourselves to the case $\pi_1(M)=0$; the case $\pi_1(M)\neq 0$ is reduced to this case, if the universal covering $\tilde{M}$ of M is compact, otherwise one works with elements in the fundamental group $\pi_1(M)$.

For $\pi_1(M)=0$ there exists a k, $2\leqq k\leqq n$, and a map $h\colon S^k\to M$ giving a nontrivial homology class in $H_k(M)=H_k(M, \mathbf{Z}_2)$. This induces a map $h_*\colon H_*(C(S^k), C^0(S^k))\to H_*(\Pi(M), \Pi^0(M))$ which is injective for three pairwise subordinated homology classes

$$y_{i(k-1)}\in H_{i(k-1)}(C(S^k), C^0(S^k)),\qquad i=1, 2, 3,$$

and which has the property that their images can be represented by cycles possessing local cross sections in $\Lambda(M)$. So we can apply the Lemma.

We give a description of the classes $y_{i(k-1)}$: First note that $C(S^k)-C^0(S^k)$ is an open $(k-1)$-disc bundle over the space $G(S^k)$ of great circles on S^k where $G(S^k)$ is isomorphic to the grassmannian $G(2, k-1)$ of 2-planes in real $(k+1)$-space. Let $w^{k-1} \in H^{k-1}(G(2, k-1))$ be the Whitney class of this bundle and $u^{k-1} \in H^{k-1}(C(S^k), C^0(S^k))$ the Thom class of the Thom space $C(S^k)$ mod $C^0(S^k)$ of the bundle. One finds that $(u^{k-1})^3 = u^{k-1} \cup (w^{k-1})^2 \in H^{3k-3}(C(S^k), C^0(S^k))$ is $\neq 0$. Let y_{3k-3} be the generator of $H_{3k-3}(C(S^k), C^0(S^k))$, which is dual to $(u^{k-1})^3$, and put $y_{2k-2} = y_{3k-3} \cap w^{k-1}$ and $y_{k-1} = y_{2k-2} \cap w^{k-1}$.

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