ON THE EMBEDDABILITY AND NONEMBEDDABILITY OF CERTAIN PARALLELIZABLE MANIFOLDS[1]

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Introduction. The problem of proving nonembeddability results for differentiable manifolds has received much attention in recent years. However, with few exceptions (see, for example Hantzsche [3] and Massey [8]), the techniques used require that the tangent bundle of the manifold in question be nontrivial; thus they do not apply to parallelizable manifolds. In this note, we study the embeddability of a certain sequence of parallelizable manifolds. As a consequence, we are able to show that for any positive integer k, there are parallelizable manifolds which do not embed with codimension[2] k. In addition, we give an example of a 22-dimensional manifold M_1 with the property that, for any j, $1 \leqq j \leqq 7$, there are two embeddings of M_1 in R^{30+j} with fiber homotopically distinct normal sphere bundles.[3]

The results announced in this note follow from a detailed study of the embeddability and nonembeddability of sphere bundles over spheres. The complete proofs will appear in a subsequent paper.

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Statement of results. In what follows, all manifolds, embeddings and immersions will be C^∞ differentiable. We denote by $\tau(M)$ the tangent bundle of M and by $\theta^r = \theta^r(M)$ the trivial r-plane bundle over M. If ξ is a $(k-1)$-sphere bundle, $\hat{\xi}$ will denote the associated k-plane bundle.

Let S^{n-1} denote the $(n-1)$-sphere where $n = 2^{4q}$, $q \geqq 1$. It follows from results of Eckmann [2] and Adams [1] that S^{n-1} has exactly $8q$ independent vector fields. Thus we can find an $(n-8q-1)$-sphere bundle ξ_q over S^{n-1} with a cross section and with the property that $\hat{\xi}_q \oplus \theta^{8q-1} = \tau(S^{n-1})$. Let M_q denote the total space of ξ_q. Clearly the dimension of M_q is $2n-8q-2 = 2^{4q+1} - 8q - 2$.

The following proposition follows from Theorem IX of Kervaire

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[2] We say that M embeds in Euclidean space with codimension k if M can be embedded in R^{n+k} where $n =$ dimension M.

[3] A. Haefliger has proved the existence of an embedding of S^{11} in R^{17} with a normal sphere bundle which is not trivial but is fiber homotopically trivial (see Massey [9]).

[6] (see also Sutherland [12]).

PROPOSITION. *The manifolds M_q are parallelizable.*

Let ζ be the sphere bundle associated with $\hat{\xi}_q \oplus \theta^{8q}$ and let E_ζ be its total space. Then M_q can be embedded in E_ζ. Furthermore, since $\hat{\xi}_q \oplus \theta^{8q-1} = \tau(S^{n-1})$, ζ is the trivial $(n-1)$-sphere bundle over S^{n-1} so $E_\zeta = S^{n-1} \times S^{n-1}$. Therefore E_ζ and consequently M_q can be embedded in R^{2n-1}. In fact, we prove

THEOREM 1. *For $q>1$, M_q can be embedded in Euclidean space with codimension $8q+1$ but not with codimension $8q$.*

To show that M_q cannot be embedded with codimension $8q$, we first prove that we can choose a cross section $s: S^{n-1} \to M_q$ of ξ_q which is an embedding. In fact, we can pick s to have a normal bundle ν_s with the property that $\nu_s \oplus \theta^1 = \hat{\xi}_q$. Now suppose M_q embeds in R^{2n-2} with normal bundle ν. Using the fact that the composite embedding

$$S^{n-1} \longrightarrow M^q \longrightarrow R^{2n-2}$$

has a trivial normal bundle (see Kervaire [7]), and the fact that $\nu_s \oplus \theta^1 = \hat{\xi}_q$, we prove that $(\nu | S^{n-1}) \oplus \theta^{n-8q-2} = \tau(S^{n-1})$ where $\nu | S^{n-1}$ is the restriction of ν to $s(S^{n-1})$. However, this is impossible by the result of Adams [1] since $n-8q-2>8q$.

As an immediate consequence of Theorem 1, we have

COROLLARY 1. *For any positive integer k, there are parallelizable manifolds which cannot be embedded in Euclidean space with codimension k.*

Following Sanderson [11], we define the *divergence* of a manifold M to be $k-r$ where k is the least integer such that M can be embedded with codimension k and r is the least integer such that M can be immersed with codimension r. Since any parallelizable manifold can be immersed with codimension 1 (see Hirsch [4]), we have

COROLLARY 2. *For any integer k, there are manifolds with divergence exceeding k.*

We now turn our attention to the 22-dimensional manifold M_1.

THEOREM 2. *The manifold M_1 can be embedded in R^{30} but not in R^{28}. Furthermore, any embedding of M_1 in R^{30} has a normal sphere bundle which is not fiber homotopically trivial.*

We are unable to decide whether or not M_1 embeds in R^{29}.

The techniques used to prove Theorem 1 show that M_1 cannot be embedded in R^{28} while the methods of James-Whitehead [5] are used to prove that any embedding of M_1 in R^{30} must have a normal sphere bundle which is not fiber homotopically trivial. To show that M_1 does embed in R^{30}, we prove that there is a 6-sphere bundle η over S^{15} with $\xi \oplus \hat{\eta} = \theta^{15}$. Thus M_1 can be embedded in $S^{15} \times S^{14}$ which can be embedded in R^{30}.

In fact, it can be shown that $\hat{\eta} \oplus \theta^8 = \tau(S^{15})$ and that $\nu | S^{15} = \hat{\eta} \oplus \theta^1$ where ν is the normal bundle of the embedding of M_1 in R^{30} described above. Therefore, if we consider the composite embedding $M_1 \subset R^{30} \subset R^{37}$, its normal bundle when restricted to S^{15} is $\hat{\eta} \oplus \theta^8 = \tau(S^{15})$ which is not fiber homotopically trivial (see Milnor-Spanier [10, Theorem 2]). Furthermore, the embedding of M_1 in R^{31} described just before the statement of Theorem 1 has a trivial normal bundle. Thus we have

THEOREM 3. *For each j, $1 \leqq j \leqq 7$, there are two embeddings of M_1 in R^{30+j} with the property that the normal sphere bundle of the first is trivial while the normal sphere bundle of the second is not even fiber homotopically trivial.*

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