ON A PROBLEM OF PAPAKYRIAKOPOULOS

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Let P^* be the group generated by the symbols a, b, c_2, d_2, c_3, d_3, $\cdots$, c_g, d_g and subject to the relation

$$G^*: aba^{-1}b^{-1}c_2d_2c_2^{-1}d_2^{-1} \cdots c_g^{-1}d_g^{-1} = 1.$$

Let w^* be an element of P^* and N^* the normal subgroup the word $R^* = w^*aw^{*-1}a^{-1}$ generates in P^*.

The question whether the factor group P^*/N^* is torsionfree (has no element of finite order) arose in connection with the Poincaré conjecture [4]. I shall sketch a proof of the fact that P^*/N^* is torsionfree (a detailed proof will appear elsewhere).

An extension E of a torsionfree group H by the free cyclic group is torsionfree, so I will present P^*/N^* as such an extension. Consider the normal subgroup H^* which the symbols c_2, d_2, $\cdots$, c_g, d_g and b generate in P^*/N^*; its factorgroup is the free cyclic group generated by a, so P^*/N^* is an extension of H^* by the free cyclic group generated by a. I shall show now that H^* is torsionfree.

The presentation below of H^*, from which the required property is clearly seen, is based on the infinite set of generating symbols b_k, c_{ik}, d_{ik}, where, for every integer k,

$$b_k = a^{-k}ba^k, \qquad c_{ik} = a^{-k}c_ia^k, \qquad d_{ik} = a^{-k}d_ia^k.$$

The left-hand side of the defining relation G^* given above for P^*, written in terms of these symbols, becomes

$$G_0 = b_{-1}b_0^{-1}c_{20}d_{20} \cdots c_{g0}^{-1}d_{g0}^{-1}$$

and the conjugates $a^{-k}G^*a^k$ become

$$G_k = b_{k-1}b_k^{-1}c_{2k}d_{2k} \cdots c_{gk}^{-1}d_{gk}^{-1}.$$

The left hand side of the defining relation $R^* = 1$ given above for P^*/N^* can also be written by means of these symbols: there is an integer h for which w^*a^h can be so written (namely when w^*a^h contains the symbol a to exponent sum zero), say, in the form

$$v_0 = v(b_s, \cdots, c_{2t}, \cdots, d_{gu}, \cdots),$$

so that, if we define

$$v_k = v(b_{s+k}, \cdots, c_{2,t+k}, \cdots, d_{g,u+k}),$$

for integral k,

$$\overset{-1}{v_{-1}} = \overset{-1}{v}(b_{s-1}, \cdots, c_{2,t-1}, \cdots, d_{g,u-1})$$

is the form taken by $a(a^{-h}w^{*-1})a^{-1}$, whence

$$R^* = w^*aw^{*-1}a^{-1} = w^*a^ha(a^{-h}w^{*-1})a^{-1}$$

becomes

$$R_0 = v_0\overset{-1}{v_{-1}}$$

and $a^{-k}R^*a^k$ becomes

$$R_k = v_k\overset{-1}{v_{k-1}}.$$

Letting x_{ij} stand for the symbols c_{ij} and d_{ij}, we get the following presentation H of the subgroup H^* of P^*:

$$H = (x_{ij}, b_j; G_j, R_j, i: 2, \cdots, g, j: 0, \pm 1, \cdots).$$

If the two sets of words $(G_j, R_j, j: 0, \pm 1, \cdots)$ and $(G_j, A_j, j: 0, \pm 1, \cdots)$ generate the same normal subgroup in the free group on the symbols of H, then

$$H = (x_{ij}, b_j; G_j, A_j, i: 2, \cdots, g, j: 0, \pm 1, \cdots).$$

I will pick the set A to suit my purpose.

If P^*/N^* has torsion, so does the group H [1]. I shall express H as the free product of isomorphic groups H_r, $r: 0, \pm 1, \cdots$, with a free subgroup amalgamated between H_r and H_{r+1}. If H has torsion, so does H_r [1; 2]. The latter will however prove torsionfree.

Using combinatorial arguments, it can be shown that there is an A_0 (cyclically reduced, i.e. not of the form zBz^{-1} for $z \neq 1$) with the following properties:

1. If A_0 contains any b_j-symbol, then it contains only b_0;
2. A_0 actually contains some x_{ij}-symbols, and either only $j=0$ occurs for these, or else A_0 contains an x_{ij} with j at most zero and also an $x_{i'j'}$ with j' at least one;
3. A_0 is not a formal power B^k unless $k = \pm 1$.

Suppose that among the subscripts j of the x_{ij}, $i: 2, \cdots, g$ in A_0 the least is u, the largest v. Then, by property 2, above, either $u=v=0$, or $u \leqq 0$, $v \geqq 1$.

Let r be some integer. Define the groups H_r and S_r as follows:

$$H_r = (x_{i,j+r}, b_r; A_r, i: 2, \cdots, g, j: u, \cdots, v)$$

and S_r the subgroup of H_r generated by a set of elements x_r in H_r such that when $u \leqq 0 < v$ for A_0

$$x_r = (x_{2,u+r+1}, x_{3,u+r+1}, \cdots, x_{g,u+r+1}, \cdots, x_{g,v+r}, b_r^{-1})$$

and when $u = v = 0$, x_r is the last element b_r^{-1} above.

Similarly, define the groups

$$H_{r+1} = (x_{i,j+r+1}, b_{r+1}; A_{r+1}, i: 2, \cdots, g, j: u, \cdots, v)$$

and T_r the subgroup of H_{r+1} generated by a set of elements X_r in H_{r+1} such that when $u \leqq 0 < v$ for A_0

$$X_r = (x_{2,u+r+1}, x_{3,u+r+1}, \cdots, x_{g,u+r+1}, \cdots, x_{g,v+r},$$
$$b_{r+1}^{-1} c_{2,r+1} d_{2,r+1} c_{2,r+1}^{-1} d_{2,r+1}^{-1} c_{3,r+1} \cdots d_{g,r+1}^{-1})$$

and, when $u = v = 0$, X_r is the last element listed above.

According to the Freiheitssatz [**3**], in a group on one cyclically reduced defining relation, every subset of the generating symbols gives rise to a free subgroup provided not every symbol present in that defining relation occurs in the set in question. This condition holds for the symbols of x_r in H_r and the symbols X_r in H_{r+1}. Therefore, properties 1 and 2 of A_0, inherited by A_r and A_{r+1}, imply that the subgroups S_r and T_r are free groups isomorphic under the mapping that associates the two sets of elements x_r and X_r in the order given above.

Because of property 3 of A_0, inherited by A_r, the group H_r is torsionfree [**1**], and so is the free product of H_r and H_{r+1} with amalgamation of S_r and T_r and, finally, the free product of all H_r with the (infinitely many) amalgamations of S_r and T_r, $r = 0, \pm 1, \pm 2, \cdots$.

Inspection of the defining relations $G = 1$ shows that the last named group is H. Thus P^*/N^* is torsionfree.

Bibliography

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