AN INFINITE PRIMARY ABELIAN GROUP WITHOUT PROPER ISOMORPHIC SUBGROUPS[1]

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In a recent article[2] Beaumont and Pierce considered the problem of determining those abelian groups which have proper isomorphic subgroups. For primary groups they obtained the following: if the cardinality of a reduced primary abelian group G is either $\aleph_0$ or greater than $2^{\aleph_0}$, then G has a proper isomorphic subgroup. The question remained whether or not this result holds for all infinite reduced primary groups. Here we answer this question by showing that *there exists an infinite primary abelian group without elements of infinite height which has no proper isomorphic subgroups.*

Throughout this note the usual terminology and notation[3] is employed with possibly the following exception: if x is an element of a group then $[x]$ denotes the cyclic subgroup generated by x. The cardinality of a set M is denoted by $|M|$.

Let p be a fixed but arbitrary prime. For each $k=1, 2, 3, \cdots$ let $C_k=[g_k]$ be a cyclic group of order p^k generated by the element g_k. Let C be the torsion subgroup of the complete direct sum of the groups C_k $(k=1, 2, \cdots)$. The elements x in C are then countable sequences

$$x = (x_1, x_2, \cdots, x_k, \cdots)$$

where $x_k \in C_k$ and the orders of the x_k's are uniformly bounded; the element x_k is called the kth *component* of x. Also C is a *closed* p-group in that every Cauchy sequence in C has a limit.[4] Set $h_k=p^{k-1}g_k$, and define the elements b_k and c_k $(k=1, 2, \cdots)$ by

$$b_k = (0, \cdots, 0, g_k, 0, \cdots),$$
$$c_k = (0, \cdots, 0, h_k, 0, \cdots) = p^{k-1}b_k.$$

Each c_k is then an element of order p. Moreover if x is an element in C of order p, then there is a sequence of integers $u_1, u_2, u_3, \cdots$ such that

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[2] R. A. Beaumont and R. S. Pierce, *Partly transitive modules and modules with proper isomorphic submodules*, Trans. Amer. Math. Soc. **91** (1959), 209–219.

[3] See, for example, L. Fuchs, *Abelian groups*, Budapest, 1958.

[4] Ibid., p. 114.

$$x = (u_1h_1, u_2h_2, u_3h_3, \cdots),$$

and x is the limit in C of the Cauchy sequence

$$u_1c_1,\ u_1c_1 + u_2c_2,\ u_1c_1 + u_2c_2 + u_3c_3, \cdots.$$

Let $\sigma = \{s_1, s_2, s_3, \cdots\}$ be a sequence of elements in C of order p such that s_k has height at least $k-1$ for each $k=1, 2, \cdots$. If $x=(u_1h_1, u_2h_2, u_3h_3, \cdots)$ is any element in C of order p, then the elements

$$u_1s_1,\ u_1s_1 + u_2s_2,\ u_1s_1 + u_2s_2 + u_3s_3, \cdots$$

form a Cauchy sequence and hence have a limit in C; this limit we denote by x^σ. Furthermore, if σ has the property, that for *no* integers u and v is it true that $s_k = uc_k$ for all $k \geqq v$, then σ is called *distinguished.*

We now observe the following: if $\sigma = \{s_1, s_2, s_3, \cdots\}$ is distinguished, then there exists a set T_σ of elements of order p such that $|T_\sigma| = 2^{\aleph_0}$ and such that for every pair of distinct elements $x, y \in T_\sigma$,

$$[x] + [x^\sigma] + [y] + [y^\sigma] = [x] \oplus [x^\sigma] \oplus [y] \oplus [y^\sigma].$$

In constructing T_σ there are three cases to consider. The first: there is an infinite sequence of integers $n_1, n_2, n_3, \cdots$ such that s_{n_k} has height greater than $n_k - 1$ for each $k=1, 2, \cdots$ (and hence each s_{n_k} has its first n_k components zero). In this case we may assume that $n_{k+1}-1$ is greater than the height of s_{n_k} for each $k=1, 2, \cdots$. Let $S \subseteq \{n_2, n_3, n_4, \cdots\}$, and define an element $x(S)$ by

$$x(S)_i = \begin{cases} 0 & \text{if } i \neq n_1 \text{ and } i \notin S, \\ h_i & \text{if } i = n_1 \text{ or } i \in S, \end{cases}$$

where $x(S)_i$ denotes the ith component of $x(S)$. Then the set $T_\sigma = \{x(S) \mid S \subseteq \{n_2, n_3, n_4, \cdots\}\}$ is easily seen to have the desired properties. The second case is: there is an infinite sequence of integers $n_1, n_2, n_3, \cdots$ such that s_{n_k} has height $n_k - 1$ and has *more* than one nonzero component for each $k=1, 2, \cdots$. In this case we may assume that n_{k+1} is greater than the index of the second nonzero component of s_{n_k} for each k. If for every $S \subseteq \{n_2, n_3, n_4, \cdots\}$ an element $x(S)$ is defined as in the preceding case, then

$$T_\sigma = \{x(S) \mid S \subseteq \{n_2, n_3, n_4, \cdots\}\}$$

again has the desired properties. The remaining case is the following: there is an integer m such that s_i has height $i-1$ and exactly one nonzero component for each $i \geqq m$. In this case, since σ is distinguished, there are two infinite disjoint sequences of integers $n_1, n_2,$

$n_3, \cdots$ and $m_1, m_2, m_3, \cdots$, and two integers u and v such that $u \not\equiv v \pmod{p}$ and such that for each k we have $n_k, m_k \geqq m$, $s_{n_k} = uc_{n_k}$, and $s_{m_k} = vc_{m_k}$. Let $S \subseteq \{2, 3, 4, \cdots\}$ and define an element $x(S)$ by

$$x(S)_i = \begin{cases} 0 & \text{if } i \neq n_k \text{ and } i \neq m_k \text{ for all } k \in S \cup \{1\}, \\ h_i & \text{if } i = n_k \text{ or } i = m_k \text{ for some } k \in S \cup \{1\}, \end{cases}$$

where $x(S)_i$ denotes the ith component of $x(S)$. Then the set $T_\sigma = \{x(S) \mid S \subseteq \{2, 3, 4, \cdots\}\}$ has the required properties.

It is clear that there are $2^{\aleph_0}$ distinct distinguished sequences. Let Ω be the first ordinal of cardinality $2^{\aleph_0}$, and let

$$\alpha \leftrightarrow \sigma_\alpha$$

be a one-one correspondence between the distinguished sequences and the ordinals $\alpha < \Omega$. Let c be any element of order p in C which is not in the subgroup generated by the elements $c_1, c_2, c_3, \cdots$.

We now construct a subgroup $P_\alpha \subseteq \{x \in C \mid px = 0\}$ for each $\alpha < \Omega$ with the following properties:

(i) $P_\alpha \supseteq P_\beta \supseteq \{c_1, c_2, c_3, \cdots\}$ for all $\beta \leqq \alpha$;

(ii) $|P_\alpha| \leqq |\alpha| \aleph_0$;

(iii) $c \notin P_\alpha$;

(iv) there exists an element $x_\alpha \in P_\alpha$ such that $c - x_\alpha^{\sigma_\alpha} \in P_\alpha$.

Suppose P_β has been constructed for each $\beta < \alpha$. Set

$$P'_\alpha = \bigcup_{\beta < \alpha} P_\beta + [c_1] + [c_2] + [c_3] + \cdots.$$

It then follows that $|P'_\alpha + [c]| \leqq |\alpha| \aleph_0 < 2^{\aleph_0}$, and since there are $2^{\aleph_0}$ pairwise disjoint subgroups of form

$$[x] \oplus [x^{\sigma_\alpha}], \qquad x \in T_{\sigma_\alpha},$$

there exists an element $x_\alpha \in T_{\sigma_\alpha}$ such that

$$([x_\alpha] \oplus [x_\alpha^{\sigma_\alpha}]) \cap (P'_\alpha + [c]) = 0.$$

If $c \in P'_\alpha + [x_\alpha] + [c - x_\alpha^{\sigma_\alpha}]$, then there are integers u, v and an element $y \in P'_\alpha$ such that

$$c = y + ux_\alpha + v(c - x_\alpha^{\sigma_\alpha}).$$

Consequently

$$ux_\alpha - vx_\alpha^{\sigma_\alpha} \in P'_\alpha + [c],$$

whence $u \equiv v \equiv 0 \pmod{p}$. But then $c = y \in P'_\alpha$, an impossibility. Therefore if we set

$$P_\alpha = P'_\alpha + [x_\alpha] + [c - x_\alpha^{\sigma_\alpha}],$$

then $c \notin P_\alpha$, and it follows that P_α satisfies (i)–(iv).

Having constructed the sequence P_α $(\alpha < \Omega)$, define the subgroup P by

$$P = \bigcup_{\alpha<\Omega} P_\alpha.$$

Let G be a pure subgroup of C such that

$$G \supseteq \{b_1, b_2, b_3, \cdots\}$$

and

$$\{x \in G \mid px = 0\} = P.$$

Suppose f is an isomorphic mapping of G into itself. Then the sequence σ_f of the elements

$$f(c_1), f(c_2), f(c_3), \cdots$$

is a sequence of elements of order p, and clearly each $f(c_k)$ has height at least $k-1$. Suppose σ_f is not distinguished. Then there are integers u and v such that $f(c_k) = uc_k$ for all $k \geqq v$. Let Q be the subgroup of those elements in P which are limits of Cauchy sequences from the subgroup generated by $c_v, c_{v+1}, c_{v+2}, \cdots$, that is, Q is the subgroup of P consisting of these elements whose first $v-1$ components are zero. Observe that $f(x) = ux$ for all $x \in Q$, and since u is necessarily relatively prime to p, we have $f(Q) = Q$. Let H be the subgroup of G consisting of those elements whose first $v-1$ components are zero. Since G contains the elements $b_1, b_2, b_3, \cdots$, H is a pure subgroup of G. Moreover

$$\{x \in H \mid px = 0\} = Q = \{y \in f(H) \mid py = 0\},$$

and it follows that $f(H)$ is a pure subgroup of G. Let x be any element of order p in G. Then there are integers $w_1, \cdots, w_{v-1}$ and an element $y \in Q$ such that

$$x = w_1c_1 + \cdots + w_{v-1}c_{v-1} + y,$$

and since $f(H)$ is pure we infer that $G/f(H)$ has rank $v-1$. Suppose there is an element $z \in G$ such that $x \equiv p^{v-1}z \pmod{f(H)}$. Then z has order p^v modulo $f(H)$, and since $f(H)$ is pure we may assume that the actual order of z is p^v. The order of $p^{v-1}z$ is therefore p, and clearly the first $v-1$ components of $p^{v-1}z$ are zero. Hence $p^{v-1}z \in Q$, and as $x - p^{v-1}z \in Q$ we conclude that $x \in Q \subseteq f(H)$. Thus every element in $G/f(H)$ of order p has height at most $v-2$, and it follows that

$G/f(H)$ is finite. Consequently there exists a *finite* subgroup F such that

$$G = F \oplus f(H).$$

Furthermore

$$f(G) = F' \oplus f(H)$$

where $F' = F \cap f(G)$. Since $f(G) \cong G$, it follows that $F \cong F'$; whence $F = F'$. Hence $f(G) = G$, and f is an automorphism when σ_f is not distinguished. Suppose then that σ_f is distinguished. Then there is an ordinal $\alpha < \Omega$ such that $\sigma_\alpha = \sigma_f$. If the element x_α has the form $x_\alpha = (u_1h_1, u_2h_2, u_3h_3, \cdots)$, then x_α is the limit in C, and hence the limit in G since G is pure, of the Cauchy sequence

$$u_1c_1, \; uc_1 + u_2c_2, \; u_1c_1 + u_2c_2 + u_3c_3, \cdots,$$

and consequently $f(x_\alpha)$ is the limit in $f(G)$ (and hence the limit is C) of the Cauchy sequence

$$u_1f(c_1), \; u_1f(c_1) + u_2f(c_2), \; u_1f(c_1) + u_2f(c_2) + u_3f(c_3), \cdots.$$

But this limit is $x_\alpha^{\sigma_f} = x_\alpha^{\sigma_\alpha}$, whence $f(x_\alpha) = x_\alpha^{\sigma_\alpha} \in G$. Since $c - x_\alpha^{\sigma_\alpha} \in G$ it follows that $c \in G$, a contradiction. Therefore σ_f cannot be distinguished. We conclude that every isomorphic mapping of G into itself is an automorphism, and the proof is complete.

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